

On the Aloha throughput-fairness tradeoff

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Abstract

A well-known inner bound of the stability region of the slotted Aloha protocol on the collision channel with n users assumes worst-case service rates (all user queues non-empty). Using this inner bound as a feasible set of achievable rates, a characterization of the throughput–fairness tradeoff over this set is obtained, where throughput is defined as the sum of the individual user rates, and two definitions of fairness are considered: the Jain-Chiu-Hawe function and the sum-user α -fair (isoelastic) utility function. This characterization is obtained using both an equality constraint and an inequality constraint on the throughput, and properties of the optimal controls, the optimal rates, and the fairness as a function of the target throughput are established. A key fact used in all theorems is the observation that all contention probability vectors that extremize the fairness functions take at most two non-zero values.

Index Terms

multiple access; random access; Aloha; stability; throughput-fairness tradeoff; Jain fairness; α -fair; proportional fair.

I. INTRODUCTION

We investigate the throughput–fairness tradeoff for the slotted Aloha medium access control (MAC) protocol [1], [2] serving n users contending on a shared collision channel. Throughput–fairness tradeoffs naturally arise in settings of shared access to a constrained resource, where maximum use of the resource is at odds with fair access to the resource, on account of the inefficiency incurred in resource contention. In the setting of Aloha, this incurred inefficiency takes the form of wasted slots in which either no user contends (idle) or multiple users contend (collision). Trivially, maximum throughput of one successful packet per time slot is achieved by the unfair allocation granting one user access and shutting out all other users, while the maximally fair allocation granting each user equal access achieves a throughput that decays to zero in the number of users. Our focus is on characterizing the tradeoff connecting these two extreme points.

Although modern MAC protocols in use today are far more complex and more sophisticated than Aloha, many of them nonetheless retain at their core the notion of random access, which is the defining characteristic of Aloha. It is therefore natural, in our opinion, to first analyze the throughput–fairness tradeoff in random access in the canonical setting of slotted Aloha before seeking to characterize such tradeoffs under more complicated protocols.

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One difficulty precluding this goal from being achieved is that the stability region for slotted Aloha on the collision channel remains unknown, in spite of 40+ years of effort. Because of this, we employ a well-known inner bound on the stability region, obtained by assuming each of the user’s queues is nonempty, thereby yielding a worst-case effective service rate seen by each user. This inner bound is known to be tight for all special cases for which the stability region of slotted Aloha is known. Even with this simplifying assumption, however, the throughput–fairness problem is still nontrivial on account of the fact that the inner bound cannot be described explicitly. Rather, the inner bound is given as the image of the function mapping contention probability vectors (controls) to (worst-case) packet transmission rates, over the set of all possible controls.

A. Related work

The throughput–fairness tradeoff literature is quite large and diverse, stemming from its relevance to a wide variety of disciplines, including queueing theory, communication networks, optimization, and economics. As such, we restrict our discussion to only the most pertinent prior work. Specifically, we summarize prior work on each of the two fairness metrics used in this paper, namely, the Jain-Chiu-Hawe function and the α -fair utility function.

The Jain-Chiu-Hawe fairness measure [3], hereafter simply Jain’s fairness, measures the fairness of an n -vector $\mathbf{x} = (x_1, \dots, x_n)$, representing in our context the vector of user rates, as the normalized distance from \mathbf{x} to the “all-rates-equal” ray passing from the origin through the point $\mathbf{1}$. This metric has been widely adopted, e.g., [4], [5].

The α -fair parameterized family of utility functions was introduced to the networking community in [6], but is nearly identical to the classic isoelastic utility function in economics [7]. The α -fair family of utility functions has found profitable use in characterizing throughput–fairness tradeoffs and resource allocation policies in wired and wireless networks, and in that sense may be viewed as part of the larger body of work termed network utility maximization (NUM), e.g., [8], [9], [10], [11]. The basic concept in NUM is to associate with each user a utility (often assumed to be concave increasing) that depends upon the resources allocated to the user, and seek a feasible resource allocation that maximizes the sum-user utility. In essence, the concavity of the utility function captures the law of diminishing returns for each user, and thus optimizing sum utility over all feasible allocations yields a solution that is “fair” in the sense that all users enjoy a common marginal utility. Returning to α -fair utility functions, the parameter $\alpha \geq 0$ controls the “concavity” of the utility function, where $\alpha = 0$ corresponds to a linear utility function (no diminishing returns), $\alpha = 1$ is a logarithmic utility function (so-called proportional fair utility), and as $\alpha \rightarrow \infty$ the utility-optimal resource allocation is the so-called max-min fair allocation. Given this, it is natural to think that increasing α would trade sum-user throughput for fairness, although recent work [12], [13], [14], [15] has identified counter-examples.

Recent work has addressed throughput–fairness tradeoffs using both these fairness measures in the context of *downlink* scheduling [15], [5]. In contrast, our focus is on *uplink*, and this fundamental difference limits the applicability of many of the results in [15], [5] to our setting. An axiomatic approach to fairness is given in [16], with an insightful discussion contrasting Jain’s fairness and α -fairness.

TABLE I
SUMMARY OF RESULTS

§#/Result	Title/Description
§II	Model and problem statement
Lem. 1	“All-rates” equal ray’s geometric and algebraic properties
§III	Properties of optimal controls
Prop. 1	Schur-concavity of fairness measures in rate space
Prop. 2	Majorization properties under throughput constraint
Cor. 1	Sufficiency to optimize over $\partial\Lambda$ (or $\partial\mathcal{S}$ in control space)
Prop. 3	Properties of controls in $\overline{\partial\mathcal{S}_2}$ under throughput constraint
Prop. 4	Sufficiency to optimize over the restricted set in Def. 1
§IV	Jain-Chiu-Hawe fairness tradeoff
Prop. 5	T-F tradeoff under Jain’s fairness when $n = 2$
Prop. 6	Monotonicity properties of the Jain’s objective over $\partial\mathcal{S}_2$
Thm. 1	T-F tradeoff under Jain’s fairness for general $n \geq 2$
Thm. 2	No change under throughput inequality constraint
Alg. 1	Incremental plotting of T-F tradeoff for a sequence of n ’s
Thm. 3	Properties of the Jain T-F tradeoff
§V	α-fair network utility maximization ($\alpha \geq 1$)
Prop. 7	T-F tradeoff under α -fairness when $n = 2$
Prop. 8	Monotonicity property of the α -fair objective over $\partial\mathcal{S}_2$
Thm. 4	T-F tradeoff under α -fairness for general $n \geq 2$
Thm. 5	Change under throughput inequality constraint
Thm. 6	Properties of the α -fair T-F tradeoff

B. Outline and contributions

The primary contribution of this paper is a characterization of the throughput–fairness (T-F) tradeoff for n users employing slotted Aloha on a collision channel. This is done through six theorems:

- Theorem 1 (2) gives the T-F tradeoff under Jain’s fairness with a throughput equality (inequality) constraint and Theorem 3 gives properties of the optimal controls, optimal rates, and the T-F tradeoff itself.
- Theorem 4 (5) gives the T-F tradeoff under α -fairness with a throughput equality (inequality) constraint, and Theorem 6 gives properties of the optimal controls, optimal rates, and the T-F tradeoff itself.

This rest of the paper is organized as follows. The model and problem statement are introduced in §II, while §III contains results common to both fairness measures. Building upon §III, the next two sections (§IV, §V) address the Aloha throughput-fairness tradeoff under Jain’s and α -fairness respectively. Finally §VI offers a brief conclusion. Three appendices follow the references, holding long proofs from §III, §IV, and §V respectively. Table I lists all the results in the paper, and Table II provides general notation.

TABLE II
GENERAL NOTATION

Symbol	Meaning
n	number of users; default vector length
$[n]$	positive integers up to n
\mathbf{x}	vector of user arrival rates
\mathbf{p}	vector of user contention probabilities
$\mathbf{x}(\mathbf{p})$	worst case service rates under control \mathbf{p} (2)
$\mathbf{u} = \frac{1}{n}\mathbf{1}$	uniform contention probability vector
\mathbf{m}	rate vector for $\mathbf{p} = \mathbf{u}$ (§II-D)
\mathbf{e}_i	unit vector with 1 in position $i \in [n]$
$d(\mathbf{x}, \mathbf{y})$	Euclidean distance between \mathbf{x} and \mathbf{y}
Λ	Aloha stability region inner bound (1)
$\partial\Lambda$	the boundary of the set Λ (3)
\mathcal{S}	closed standard unit simplex (§II-C)
$\partial\mathcal{S}$	probability vectors (4); efficient controls, c.f., (3)
$T(\mathbf{x})$	sum-user throughput of \mathbf{x} (5)
$F(\mathbf{x})$	fairness measure of \mathbf{x} : F_J (7) or F_α (8)
$\{\theta_t\}_{t=1}^n$	critical throughputs (6)
$\mathcal{V}(\mathbf{p})$	the set of non-zero values in \mathbf{p} (Def. 1)
$\mathbf{p}(p_s, k, n')$	restricted control vectors in Def. 1
$\partial\mathcal{S}_1$	efficient controls with $ \mathcal{V}(\mathbf{p}) = 1$ (Def. 1)
$\partial\mathcal{S}_2$	efficient controls with $ \mathcal{V}(\mathbf{p}) = 2$ (Def. 1)
$\partial\mathcal{S}_{1,2}$	efficient controls with $ \mathcal{V}(\mathbf{p}) \in \{1, 2\}$ (Def. 1)
α	parameter in α -fair utility functions (9)
θ	target throughput
$F^*(\theta)$	optimized fairness given target throughput θ

II. MODEL AND PROBLEM STATEMENT

This section is divided into the following subsections: an introduction of some general notation in §II-A, a discussion of the Aloha protocol and the collision channel in §II-B, definition of the Aloha stability region Λ_A and its inner bound Λ in §II-C, and the definitions of throughput and fairness in §II-D.

A. General notation

All vectors are lowercase and bold and are by default of length n . Inequalities between two vectors are understood to hold component-wise. We write $[n]$ to denote $\{1, \dots, n\}$ for $n \in \mathbb{N}$. The unit vector with a one in position i is denoted \mathbf{e}_i , for $i \in [n]$. The all-one vector is denoted by $\mathbf{1}$, the uniform distribution $\frac{1}{n}\mathbf{1}$ is denoted \mathbf{u} , and the all-zero vector is denoted by $\mathbf{0}$. Euclidean distance is denoted $d(\mathbf{x}, \mathbf{y})$. Cardinality of a set \mathcal{V} is denoted $|\mathcal{V}|$. We sometimes write \bar{z} to denote $1 - z$. Table II lists frequently used notation; additional notation will be explained at first use.

B. The Aloha protocol and the collision channel

Recall a MAC protocol specifies a mechanism to coordinate competing users' access to the shared channel; we consider the finite-user slotted Aloha MAC protocol operating on a collision channel. The protocol parameters are $(n, \mathbf{x}, \mathbf{p})$, where *i*) $n \in \mathbb{N}$ is the number of users, *ii*) $\mathbf{x} \in \mathbb{R}_+^n$ is an n -vector denoting the independent arrival rates of users' data packets, which we henceforth call the *rate* vector, and *iii*) $\mathbf{p} \in [0, 1]^n$ is an n -vector indicating the user contention (or channel access) probabilities, which we henceforth call the *control* vector. Each user has an associated packet queue that can hold an infinite number of packets, stored in order of arrival. Each packet will be removed from the queue if and only if it has just been successfully transmitted. The channels are error-free. Time is slotted and synchronized. At the beginning of each time slot, every user with a non-empty queue, say user $i \in [n]$, contends for channel access to the common base station by transmitting its head-of-line packet with a fixed probability p_i , independent of anything else. The collision channel assumption means the state of the channel in each time slot may be classified as *i*) *idle* (no one attempts to transmit, either because of having an empty queue or electing not to transmit), *ii*) *collision* (more than one user transmits, and all attempted transmissions fail), or *iii*) *success* (precisely one user transmits, and this attempted transmission succeeds). This ternary feedback is error-free and instantaneous at the end of each time slot.

C. The stability region Λ_A and its inner bound Λ

An important yet still open problem is the queueing-theoretic *stability region* (also called the *network layer capacity region* [17, pp. 28]) of this model, denoted Λ_A (A for Aloha), which contains all arrival rate vectors \mathbf{x} that can be stabilized by the protocol, i.e., for each $\mathbf{x} \in \Lambda_A$ there exists a control vector \mathbf{p} that stabilizes each of the n queues. The stability region is open even for the case of independent arrival process and $n > 2$ users. A summary of the history of this problem is provided in [18], with compelling recent work including [19], [20] among others.

As Λ_A is unknown, we employ a suitable *inner bound* on Λ_A as a *proxy* for the stability region of slotted Aloha. This inner bound, denoted Λ below, has been proved to coincide with the exact stability region for all special cases for which the stability region is known ([21], [22]), and has been conjectured ([23, §V], [18, §V Thm. 2]) to in fact *be* the stability region, Λ_A . The set Λ is defined as:

$$\Lambda \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^n : \exists \mathbf{p} \in [0, 1]^n : x_i \leq p_i \prod_{j \neq i} (1 - p_j), \forall i \in \{1, \dots, n\} \right\}. \quad (1)$$

The expression $p_i \prod_{j \neq i} (1 - p_j)$ is the worst-case service rate for user i 's queue, namely the service rate assuming all other users have non-empty queues and thus all users are eligible for channel contention. In particular, user i 's transmission is successful in such a time slot if user i elects to contend (with probability p_i) and each other user $j \neq i$ does not contend (each with independent probability $1 - p_j$). Clearly, Λ is an inner bound, since an arrival rate that is stabilizable under the worst-case service rate is certainly stabilizable under a better service rate. It may be shown [24, §II, Prop. 2] that an equivalent definition of Λ is to change all the inequalities to equality, i.e., $\mathbf{x} \in \Lambda$

if and only if there exists a $\mathbf{p} \in [0, 1]^n$ for which $\mathbf{x} = \mathbf{x}(\mathbf{p})$, where

$$x_i(\mathbf{p}) \equiv p_i \prod_{j \neq i} (1 - p_j), \quad i \in [n]. \quad (2)$$

We refer to such a \mathbf{p} as a (*critical compatible*) *control* for \mathbf{x} .¹ Based on the above definition of Λ , testing whether or not a candidate \mathbf{x} is or is not in Λ is equivalent to the solvability of $\mathbf{x} = \mathbf{x}(\mathbf{p})$ over $\mathbf{p} \in [0, 1]^n$. The definition of Λ is therefore *implicit*, in the sense that testing membership $\mathbf{x} \in \Lambda$ requires establishing the existence (or not) of a suitable control \mathbf{p} . When addressing throughput–fairness tradeoffs we will be optimizing an objective function over Λ , which thus becomes the *feasible set* for the optimization. The implicit characterization of Λ is what makes the corresponding throughput–fairness tradeoff optimization problem non-trivial. The natural solution, which we employ, is to make $\mathbf{p} \in [0, 1]^n$ the optimization variable, thereby requiring the corresponding nonlinear compositions on both the throughput and fairness functions, i.e., $T(\mathbf{x}(\mathbf{p}))$ and $F(\mathbf{x}(\mathbf{p}))$, defined below. To emphasize this distinction, we refer to \mathbf{x} as a rate vector in *rate space*, and \mathbf{p} as a control vector in *control space*.

The boundary of Λ in \mathbb{R}_+^n is denoted $\partial\Lambda$ and is characterized [25] as

$$\partial\Lambda = \left\{ \mathbf{x} \in \mathbb{R}_+^n : \exists \mathbf{p} \in \partial\mathcal{S} : x_i = p_i \prod_{j \neq i} (1 - p_j), \quad \forall i \in \{1, \dots, n\} \right\}, \quad (3)$$

where $\mathcal{S} \equiv \{\mathbf{z} \in \mathbb{R}_+^n : \sum_{i=1}^n z_i \leq 1\}$ denotes the “standard” unit simplex, and its “face”, denoted

$$\partial\mathcal{S} \equiv \{\mathbf{z} \in \mathbb{R}_+^n : \sum_{i=1}^n z_i = 1\}, \quad (4)$$

is the set of *probability vectors* on $[n]$. Thus, Pareto efficient throughputs, i.e., $\mathbf{x} \in \partial\Lambda$, are achieved by and only by controls that are probability vectors, i.e., $\mathbf{p} \in \partial\mathcal{S}$. For this reason, we call $\partial\mathcal{S}$ the set of *efficient controls*.

It may be helpful to visualize Λ and its boundary $\partial\Lambda$ using Fig. 2 (§IV-A) for the $n = 2$ case, where they are shown as the light blue shaded area and the brown curve respectively. In addition, the following lemma (the proof of which is straightforward and is omitted), used in some proofs, is relevant to Λ in that it implies: *a*) geometrically, the ray from the origin through $\mathbf{1}$ (the “all-rates equal” ray) resides inside Λ until it hits the boundary $\partial\Lambda$ at $\mathbf{x} = \frac{\theta_n}{n} \mathbf{1}$ (see (6) and the discussion below), shown in Fig. 2 as the black dot, and *b*) there only exist(s) two (one) control(s) \mathbf{p} for any rate vector \mathbf{x} on this ray segment that lies inside (on the boundary of) Λ , in the sense of (2).

Lemma 1: Let an integer $n \geq 2$ be given. The function $p(1-p)^{n-1}$ for $p \in [0, 1]$ is increasing when $p \in [0, 1/n]$ and decreasing when $p \in [1/n, 1]$, with the maximum $\frac{1}{n} \left(1 - \frac{1}{n}\right)^{n-1}$ attained at $p = 1/n$.

D. Throughput and two fairness measures

The sum-user throughput of any rate vector $\mathbf{x} \in \Lambda$ is defined as:

$$T(\mathbf{x}) \equiv \sum_{i=1}^n x_i. \quad (5)$$

¹More generally, we define a *compatible control* for \mathbf{x} as a control vector \mathbf{p} for which $\mathbf{x} \leq \mathbf{x}(\mathbf{p})$. In this paper we only employ critical compatible controls, and as such we often refer to \mathbf{p} satisfying $\mathbf{x} = \mathbf{x}(\mathbf{p})$ simply as a *control* for \mathbf{x} .

Note $T(\mathbf{x}) \in [0, 1]$ since, by the definition of the collision channel, there is at most one successful transmission on the channel in each time slot. We define the vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ with $\theta_1 = 1$ and

$$\theta_t \equiv (1 - 1/t)^{t-1}, \quad t \in \{2, \dots, n\} \quad (6)$$

as the vector of *critical throughputs*. Observe $1 = \theta_1 > \dots > \theta_n > 1/e$. Define the rate vector $\mathbf{m} \equiv \frac{\theta_n}{n} \mathbf{1} = \mathbf{x}(\mathbf{u})$ associated with θ_n , i.e., \mathbf{m} is the rate vector for the uniform control \mathbf{u} , with corresponding throughput $T(\mathbf{m}) = \theta_n$. Geometrically, \mathbf{m} is the unique intersection of the ray from the origin through $\mathbf{1}$ (the “all-rates equal” ray) with $\partial\Lambda$.

The fairness of \mathbf{x} is denoted $F(\mathbf{x})$; we will employ the following two fairness definitions in this paper. The first, *Jain-Chiu-Hawe fairness* [3], henceforth referred to simply as Jain’s fairness and denoted $F_J(\mathbf{x})$, is a now classic means of quantifying the fairness of a resource allocation \mathbf{x} :

$$F_J(\mathbf{x}) = \frac{T(\mathbf{x})^2}{n \|\mathbf{x}\|^2}. \quad (7)$$

The Jain’s fairness function has the following properties: *i*) scale invariance, i.e., $F_J(\beta\mathbf{x}) = F_J(\mathbf{x})$ for any $\beta \in \mathbb{R}_{++}$; and *ii*) boundedness, i.e., $F_J \in [1/n, 1]$, with $F_J(\beta\mathbf{e}_i) = 1/n$ for any $i \in [n]$ and $F_J(\beta\mathbf{1}) = 1$ for any $\beta \in \mathbb{R}_{++}$.

The second fairness measure, the α -fair sum-user utility function, defined as

$$F_\alpha(\mathbf{x}) \equiv \sum_{i=1}^n U_\alpha(x_i), \quad (8)$$

for $\alpha \geq 0$, is the sum-user utility of the allocation \mathbf{x} , where the (common) per-user utility functions are defined, for $\alpha \in \mathbb{R}$, as:²

$$U_\alpha(x) = \begin{cases} \log(x), & \alpha = 1 \\ \frac{1}{1-\alpha} x^{1-\alpha}, & \alpha \neq 1 \end{cases}. \quad (9)$$

Maximization of sum-user utility over a set of feasible allocations, for any concave increasing utility function $U_\alpha(x)$, often *implicitly* enforces a throughput–fairness tradeoff. For example, the cases $\alpha = 0, 1, \infty$ have corresponding optimal solutions that maximize throughput, proportional fairness (log-utility), and max-min fairness, respectively. It is for this reason that we refer to $F_\alpha(\mathbf{x})$ as a fairness function.

Observe that under the throughput equality constraint $T(\mathbf{x}) = \theta$, the objective $F_J(\mathbf{x})$ is inversely proportional to $F_{-1}(\mathbf{x})$, i.e., $F_\alpha(\mathbf{x})$ in (8) with $\alpha = -1$, and as such maximizing $F_J(\mathbf{x})$ under $T(\mathbf{x}) = \theta$ is equivalent, in the sense of having the same extremizers, to minimizing $F_{-1}(\mathbf{x})$. Even though F_α only possesses the desirable properties of a utility function for $\alpha \geq 0$, this equivalence allows us to study extremizers of F_J and F_α ($\alpha \geq 0$) under a unified framework, as in Prop. 4 in §III.

The general throughput-fairness tradeoff for slotted Aloha, using the proxy stability region Λ as the feasible set of arrival rate vectors, is the Pareto frontier of the parametric plot $(T(\mathbf{x}), F(\mathbf{x}))$ over $\mathbf{x} \in \Lambda$. An equivalent alternate

²Note that $\lim_{\alpha \rightarrow 1} U_\alpha(x) = \pm 1/0$, i.e., is undefined, and not equal to $U_1(x) = \log x$. One way to rectify this discrepancy is to modify the definition to include a constant shift, e.g., $\tilde{U}_\alpha(x) \equiv \frac{1}{1-\alpha} (x^{1-\alpha} - 1)$, which is known as the *isoelastic utility function* in economics. As is conventional in the networking literature, we omit this constant as it has no effect on the extremizers.

formulation of the throughput–fairness tradeoff is to seek to maximize $F(\mathbf{x})$ over $\mathbf{x} \in \Lambda$ such that $T(\mathbf{x}) = \theta$, for $\theta \in (0, 1)$ a target throughput constraint. We omit $\theta = 0$ and $\theta = 1$ as target throughputs as both correspond to trivial edge cases. In fact, we will address two types of throughput constraints in this paper: *i*) a throughput equality constraint $T(\mathbf{x}) = \theta$, and *ii*) a throughput inequality constraint $T(\mathbf{x}) \geq \theta$. The equality constraint is used, as mentioned above, to characterize the throughput–fairness tradeoff, while the inequality constraint admits a natural operational interpretation: allocate “resources” as fairly as possible subject to the sum throughput exceeding a minimum requirement. As we will show, there are parameter regimes wherein these two problems are the same, and regimes where they are different.

Finally, observe that Λ , $F(\mathbf{x})$, and $T(\mathbf{x})$ are each permutation invariant, and as such any extremizer \mathbf{x}^* that maximizes fairness under a throughput constraint is permutation invariant, meaning any permutation of \mathbf{x}^* is likewise an extremizer.

Further notes about notation. Auxiliary functions (typically named as f_1 , f_2 , etc.) used in proofs are understood to be *internal* meaning a different function with the same name might be used in a different proof. The following inequality about the natural logarithm function is frequently used in the paper:

$$\log(1 + z) \leq z, \quad \text{for all } z > -1, \quad (10)$$

which is strict unless $z = 0$. Finally, we use $F^*(\theta)$ to represent the maximum fairness for a given target throughput θ , which is not to be confused with $F(\mathbf{x})$ defined in (7) and (8).

III. PROPERTIES OF OPTIMAL CONTROLS

We use the framework of majorization in §III-A to establish that it suffices to restrict the control space from $[0, 1]^n$ to the set of *efficient controls*, namely $\partial\mathcal{S}$ (4), and then use Karush-Kuhn-Tucker (KKT) conditions in §III-B to establish structural properties of those controls that extremize $F_\alpha(\mathbf{x})$ for $\alpha \in (-\infty, -1] \cup [1, \infty)$ under a throughput constraint.

A. A majorization approach

We address the Aloha T-F tradeoff problem through the lens of *majorization* [26], the origins of which are rooted in questions of fairness. Majorization defines a partial order on the set of vectors with the same length and sum of components. More precisely, \mathbf{a} is majorized by \mathbf{b} , denoted $\mathbf{a} \prec \mathbf{b}$, if $\sum_{i=1}^k a_{[k]} \leq \sum_{i=1}^k b_{[k]}$ for all $k \in [n]$, where $a_{[k]}$ is the k^{th} component of \mathbf{a} sorted in nonincreasing order. For example, the “quasi-uniform” probability vectors (in $\partial\mathcal{S}$) below are majorized as [26, pp. 9]:

$$\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \prec \left(\frac{1}{n-1}, \dots, \frac{1}{n-1}, 0\right) \prec \left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right) \prec \dots \prec (1, 0, \dots, 0). \quad (11)$$

As the above example suggests, in many contexts the statement $\mathbf{x} \prec \mathbf{y}$ may be interpreted as \mathbf{x} is *more fair* than \mathbf{y} , in the sense that the components of vector \mathbf{x} are more nearly equal than those of \mathbf{y} . It is therefore natural to try to study our T-F tradeoff within the framework of majorization. The class of Schur (concave or convex) functions are

symmetric functions that preserve majorization, i.e., F is Schur concave (convex) if $F(\mathbf{x}) \geq F(\mathbf{y})$ ($F(\mathbf{x}) \leq F(\mathbf{y})$) for all (\mathbf{x}, \mathbf{y}) such that $\mathbf{x} \prec \mathbf{y}$. The following result, taken from [16] (c.f. Thm. A. 4 in Ch. 3 of [26]), indicates the relevance of Schur concavity to our problem (note Schur concavity is preserved under summation, c.f. (8)).

Proposition 1: The Jain's fairness function (7) and α -fair utility function (9) for $\alpha \geq 0$ are Schur concave in \mathbf{x} .

Remark 1: An immediate consequence of this result is that it allows us to restrict the set of feasible controls from $[0, 1]^n$ to $(0, 1)^n$. First, observe that if there are multiple users contending with probability one, then the corresponding rate vector is $\mathbf{x} = \mathbf{0}$, and as such $T(\mathbf{x}) = 0$, meaning such points cannot achieve any target throughput $\theta \in (0, 1)$. Second, if there is a unique user, say i , with $p_i = 1$ (i.e., $p_j \in [0, 1)$ for all $j \neq i$), then $\mathbf{x} = \pi_i \mathbf{e}_i$, where $\pi_i = \prod_{j \neq i} (1 - p_j)$. But, such an \mathbf{x} majorizes every other feasible point in rate space, and thus will not maximize either of our fairness objectives.

The following result establishes two key facts. First, it suffices to consider only efficient controls, $\mathbf{p} \in \partial\mathcal{S}$, for maximizing fairness under a throughput (equality) constraint. Second, there is no majorization relationship among any two efficient controls that both satisfy the throughput constraint. Thus, majorization does not by itself solve the T-F tradeoff optimization problem.

Proposition 2: Fix the number of users n and the target throughput $\theta \in (\theta_n, 1)$. Define the hyperplane $\mathcal{H}_\theta = \{\mathbf{x} \in \mathbb{R}_+^n : T(\mathbf{x}) = \theta\}$ of rate vectors with throughput θ . Define $\Lambda_\theta = \Lambda \cap \mathcal{H}_\theta$, $\partial\Lambda_\theta = \partial\Lambda \cap \mathcal{H}_\theta$, and $\Lambda_\theta^{\text{int}} = \Lambda_\theta \setminus \partial\Lambda_\theta$ as the set of stable, stable efficient, and stable inefficient rate vectors with throughput θ , respectively. Then

- 1) for any $\mathbf{x} \in \Lambda_\theta^{\text{int}}$, there exists some $\mathbf{x}' \in \partial\Lambda_\theta$ such that $\mathbf{x}' \prec \mathbf{x}$;
- 2) for any distinct \mathbf{x}, \mathbf{x}' both in $\partial\Lambda_\theta$, it holds that $\mathbf{x} \not\prec \mathbf{x}'$ and $\mathbf{x}' \not\prec \mathbf{x}$.

The proof is found in Appendix I-A. One consequence is the following.

Corollary 1: When maximizing either Jain's fairness (7) or the α -fair objective (8) over Λ subject to a throughput equality constraint $T(\mathbf{x}) = \theta$ for $\theta \in [\theta_n, 1)$, it suffices to restrict the feasible set to the set of points on the boundary of Λ that satisfy the throughput constraint, i.e., to $\partial\Lambda_\theta$ (defined in Prop. 2). This then implies an optimal control, \mathbf{p}^* , defined in §IV-B, is in $\partial\mathcal{S}$.

This corollary follows almost immediately from Prop. 1 and Prop. 2 (item 1)) taking into account the fact that $\mathbf{p} \in \partial\mathcal{S}$ iff $\mathbf{x}(\mathbf{p}) \in \partial\Lambda$ [25]. An independent proof is given in Appendix I-A for the case of Jain's fairness, highlighting the geometric intuition behind the result.

B. Optimal controls under a throughput constraint

In this subsection we present two results that apply to both the Jain's fairness analysis in §IV and the α -fair analysis in §V. First, we define some useful restrictions of the feasible set of controls in Def. 1; this restriction is an essential component in most of our subsequent proofs. Second, in Prop. 3 we present some properties associated with the throughput constraint $T(\mathbf{x}(\mathbf{p})) = \theta$ over this restricted set. Finally, Prop. 4 establishes that the optimal controls for both fairness objectives will lie in the restricted set in Def. 1.

Definition 1: Let $\mathbf{p} \in [0, 1)^n$ be a control, and define the following:

- 1) $\mathcal{V}(\mathbf{p}) = \bigcup_{i \in [n]} \{p_i\} \setminus \{0\}$. Thus $\mathcal{V}(\mathbf{p})$ ($|\mathcal{V}(\mathbf{p})|$) denotes the set (number) of distinct nonzero values³ in \mathbf{p} .
- 2) $\partial\mathcal{S}_1 = \{\mathbf{p} \in \partial\mathcal{S} : |\mathcal{V}(\mathbf{p})| = 1\}$ denotes the set of efficient controls with exactly *one* distinct nonzero value.

Note $\partial\mathcal{S}_1$ consists of all vectors \mathbf{p} (and their permutations) of the form $p_i = 1/n'$ for $i \in [n']$ and $p_i = 0$ for $i \in \{n' + 1, \dots, n\}$, for $n' \in [n]$.

- 3) $\partial\mathcal{S}_2 = \{\mathbf{p} \in \partial\mathcal{S} : |\mathcal{V}(\mathbf{p})| = 2\}$ denotes the set of efficient controls with exactly *two* distinct nonzero values.

These two values are denoted p_s, p_l (for “small” and “large”, respectively) with $0 < p_s < p_l < 1$. Moreover, any such \mathbf{p} has a total of n' nonzero components, of which k take value p_s and $n' - k$ take value p_l , for some $k \in [n' - 1]$ and some $n' \in \{2, \dots, n\}$, and $p_s \in (0, 1/n')$. Since $\mathbf{p} \in \partial\mathcal{S}$, it follows that $kp_s + (n' - k)p_l = 1$, or equivalently,

$$p_l = p_l(p_s, k, n') \equiv \frac{1 - kp_s}{n' - k}. \quad (12)$$

We call (p_s, k, n') the three free parameters which together characterize a $\mathbf{p} \in \partial\mathcal{S}_2$, and write $\mathbf{p}(p_s, k, n')$ to denote a \mathbf{p} with those parameters. The rates associated with controls p_s, p_l are denoted x_s, x_l , respectively, with

$$\begin{aligned} x_s &= x_s(p_s, k, n') \equiv p_s(1 - p_s)^{k-1}(1 - p_l)^{n'-k} \\ x_l &= x_l(p_s, k, n') \equiv p_l(1 - p_s)^k(1 - p_l)^{n'-k-1} \end{aligned} \quad (13)$$

and it is easily shown that $x_s < x_l$.

- 4) $\partial\mathcal{S}_{1,2} = \{\mathbf{p} \in \partial\mathcal{S} : |\mathcal{V}(\mathbf{p})| \leq 2\}$ denotes the set of efficient controls with *at most two* distinct nonzero values. Because $\mathbf{p} \in \partial\mathcal{S}$ it follows that $|\mathcal{V}(\mathbf{p})| \neq 0$, and thus $\partial\mathcal{S}_{1,2} = \partial\mathcal{S}_1 \cup \partial\mathcal{S}_2$. Observe $\partial\mathcal{S}_1$ may be viewed as the limiting case of $\partial\mathcal{S}_2$ as $p_s \uparrow 1/n'$. Therefore $\partial\mathcal{S}_{1,2}$ may equivalently be defined as the closure of $\partial\mathcal{S}_2$ and thus $\mathbf{p} \in \partial\mathcal{S}_{1,2}$ may also be parameterized by (p_s, k, n') with the modification that $p_s \in (0, 1/n']$. In fact, we will use $\partial\mathcal{S}_{1,2}$ and $\overline{\partial\mathcal{S}_2}$ interchangeably with the former highlighting $|\mathcal{V}(\mathbf{p})| \in \{1, 2\}$ and the latter emphasizing p_s can take the boundary value $1/n'$.

Following the $\mathbf{p}(p_s, k, n')$ parameterization in Def. 1, we further define the following shorthands to be used:

$$\begin{aligned} r_x &= r_x(p_s, k, n') \equiv \frac{x_l}{x_s} = \frac{p_l(1 - p_s)}{p_s(1 - p_l)} \\ r_{\bar{p}} &= r_{\bar{p}}(p_s, k, n') \equiv \frac{1 - p_s}{1 - p_l}. \end{aligned} \quad (14)$$

The following proposition gives properties of the solution of the throughput equality constraint $T(\mathbf{x}(\mathbf{p})) = \theta$ over $\mathbf{p} \in \overline{\partial\mathcal{S}_2}$. Leveraging the (p_s, k, n') parameterization in Def. 1, we define (for fixed $n' \in \{2, \dots, n\}$):

$$T(p_s, k, n') \equiv T(\mathbf{x}(\mathbf{p}(p_s, k, n'))) \quad (15)$$

$$\mathcal{R}(k, n') \equiv \{T(p_s, k, n') : p_s \in (0, 1/n']\} \quad (16)$$

³ $|\mathcal{V}(\mathbf{p})|$ is the number of distinct nonzero values, not the number of indices taking nonzero values.

for $p_s \in (0, 1/n']$ and $k \in [n' - 1]$. Note $\mathcal{R}(k, n')$ is the set of achievable throughputs over $\mathbf{p} \in \overline{\partial\mathcal{S}_2}$ with fixed (k, n') , i.e., the image of $T(p_s, k, n')$ over $p_s \in (0, 1/n']$. This image is a subinterval of $[0, 1]$ on account of the continuity of $T(p_s, k, n')$ in p_s .

Proposition 3: Assume $\mathbf{p} \in \overline{\partial\mathcal{S}_2}$ is parameterized using (p_s, k, n') as in Def. 1.

- 1) Fix k, n' . The throughput $T(p_s, k, n')$ is monotone decreasing in $p_s \in (0, 1/n']$, and as such at most one $p_s \in (0, 1/n']$ will solve $T(p_s, k, n') = \theta$. This unique p_s , when it exists, is denoted by $p_s(k, n', \theta)$, and is the solution to

$$T(p_s(k, n', \theta), k, n') = \theta, \quad (17)$$

which can be expressed as an order- n' polynomial (in p_s) equation.

- 2) Now only fix n' . The range of achievable throughputs for a given k is $\mathcal{R}(k, n') = [\theta_{n'}, \theta_{n'-k})$, which is an increasing (in k) nested sequence of intervals: $\mathcal{R}(1, n') \subseteq \dots \subseteq \mathcal{R}(n' - 1, n')$.
- 3) For $\theta \in [\theta_t, \theta_{t-1})$, for some $t \in \{2, \dots, n\}$, the set of (k, n') pairs for which there exists $p_s \in (0, 1/n']$ such that $T(p_s, k, n') = \theta$ is

$$\mathcal{D}_{t,n} \equiv \bigcup_{n' \in \{t, \dots, n\}} \{(k, n') \in \mathbb{N}^2 : k \in \{n' - t + 1, \dots, n' - 1\}\}, \quad (18)$$

and is illustrated in Fig. 1 (left).

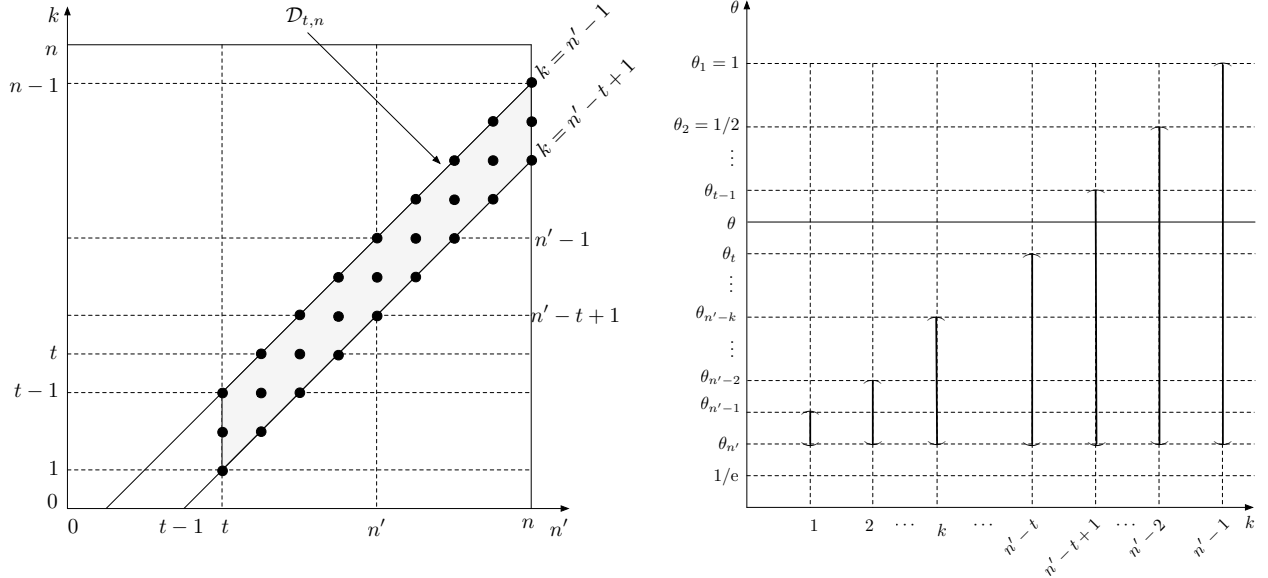


Fig. 1. **Left:** Illustration of the region $\mathcal{D}_{t,n}$ (18) (to scale, the figure shows the case $t = 4$ and $n = 12$, with the value $n' = 8$ selected on the n' axis). **Right:** Illustration that $k \in \{n' - t + 1, \dots, n' - 1\}$ is necessary and sufficient (when $n' \geq t$) for $\theta \in [\theta_t, \theta_{t-1})$ to intersect $\mathcal{R}(k, n') = [\theta_{n'}, \theta_{n'-k})$ (shown as solid vertical intervals) in (16).

The proof is in Appendix I-B. The following proposition shows that optimal controls for both the Jain's fairness and α -fair objectives will lie in the restricted set of Def. 1.

Proposition 4: Consider the following two extremization (maximization or minimization) problems, each parameterized by $\alpha \in (-\infty, -1] \cup [1, \infty)$ and $\theta \in (0, 1)$:

$$\underset{\mathbf{p} \in [0,1]^n}{\text{extremize}} F_\alpha(\mathbf{x}(\mathbf{p})) : T(\mathbf{x}(\mathbf{p})) \geq \theta, \quad \underset{\mathbf{p} \in [0,1]^n}{\text{extremize}} F_\alpha(\mathbf{x}(\mathbf{p})) : T(\mathbf{x}(\mathbf{p})) = \theta. \quad (19)$$

i) For both the inequality and equality constrained problems above, a necessary condition for \mathbf{p} to extremize (19) is $|\mathcal{V}(\mathbf{p})| \leq 2$. *ii)* For the inequality constrained problem: if an optimizer \mathbf{p}^* of (19) (left) has the property that $|\mathcal{V}(\mathbf{p}^*)| = 2$, then the throughput constraint holds with equality, i.e., $T(\mathbf{x}(\mathbf{p}^*)) = \theta$.

The proof is in Appendix I-B.

IV. JAIN-CHIU-HAWE FAIRNESS TRADEOFF

Recall from §II-D that maximizing $F_J(\mathbf{x})$ (7) under a throughput equality constraint $T(\mathbf{x}) = \theta$ is equivalent, in the sense of having the same extremizers, to minimizing $F_{-1}(\mathbf{x})$ (8), i.e., $\alpha = -1$, under the same constraint. As mentioned in §II-C, any $\mathbf{x} \in \Lambda$ may be expressed as $\mathbf{x}(\mathbf{p})$ (2) for some $\mathbf{p} \in [0, 1]^n$. Thus, an equivalent formulation of the Jain throughput–fairness optimization problem for n users with target throughput $\theta \in (0, 1)$ is:

$$\min_{\mathbf{p} \in [0,1]^n} F_{-1}(\mathbf{x}(\mathbf{p})) = \frac{1}{2} \sum_{i=1}^n x_i(\mathbf{p})^2 \text{ s.t. } T(\mathbf{x}(\mathbf{p})) = \theta. \quad (20)$$

This section is comprised of three subsections. We give: *i)* preliminary results in §IV-A, *ii)* the main results in §IV-B, and *iii)* some additional properties of the Jain throughput–fairness tradeoff in §IV-C.

A. Preliminary results

We start with the special case $n = 2$.

Proposition 5: The throughput–fairness tradeoff under Jain’s fairness metric, for $n = 2$ users, is

$$F_J^*(\theta) = \begin{cases} 1, & \theta \in (0, \frac{1}{2}] \\ \frac{\theta^2}{\theta^2 + 2\theta - 1}, & \theta \in (1/2, 1) \end{cases}. \quad (21)$$

Proof: For the $n = 2$ case we may use a direct approach (instead of solving (20)), since the set Λ may be written explicitly (i.e., parameter-free) as $\Lambda = \{\mathbf{x} \in \mathbb{R}_+^2 : \sqrt{x_1} + \sqrt{x_2} \leq 1\}$ [21], illustrated in Fig. 2.⁴ As evident from the figure, the constrained feasible set is the intersection of the throughput constraint line (for general n , a hyperplane) $\mathcal{H}_\theta = \{\mathbf{x} : x_1 + x_2 = \theta\}$ with Λ . Define the maximum fairness line $\{\mathbf{x} : x_1 = x_2\}$ (for general n , the ray emanating from the origin $\mathbf{0}$ passing through $\mathbf{1}$), on which $F_J(\mathbf{x}) = 1$. In the case of $\theta \in (0, 1/2]$, we see $\Lambda \cap \mathcal{H}_\theta$ intersects this ray, i.e., $F_J(\mathbf{x}) = 1$ is feasible. In the case of $\theta \in (1/2, 1)$, $F_J(\mathbf{x}) = 1$ is not feasible, but the fairness is easily shown to be monotone increasing on \mathcal{H}_θ as \mathbf{x} moves towards $x_1 = x_2$ (c.f., Fig. 8 in the proof of Cor. 1 in §III-A for general n), and as such, the optimal fairness is achieved at the two points for which \mathcal{H}_θ intersects $\partial\Lambda = \{\mathbf{x} \in \mathbb{R}_+^2 : \sqrt{x_1} + \sqrt{x_2} = 1\}$. These two equations together yield the solutions $(x_1^*, x_2^*) = \left(\frac{\theta \pm \sqrt{2\theta - 1}}{2}, \frac{\theta \mp \sqrt{2\theta - 1}}{2}\right)$, from which the maximum fairness may be computed to be the second expression in (21). ■

⁴ As an aside, the stability inner bound Λ is known to be exact, i.e., $\Lambda_A = \Lambda$, for the case $n = 2$ [21].

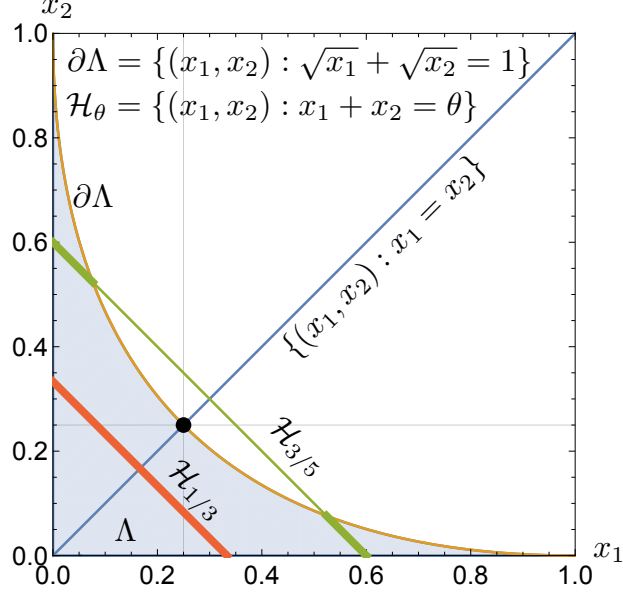


Fig. 2. Illustration of the proof of Prop. 5, the Jain throughput–fairness tradeoff for $n = 2$ users. Shown are the set Λ , its boundary $\partial\Lambda$, two throughput constraint hyperplanes \mathcal{H}_θ for $\theta \in \{1/3, 3/5\}$, and the maximum fairness line $\{(x_1, x_2) : x_1 = x_2\}$. The constrained feasible set $\Lambda \cap \mathcal{H}_\theta$ (bold line segments) intersects the maximum fairness line (on which $F_J(\mathbf{x}) = 1$) for $\theta \leq 1/2$, but not for $\theta > 1/2$.

The basic idea in establishing the Jain throughput–fairness tradeoff (Thm. 1) is to first apply Cor. 1 in §III-A to restrict the feasible set from $\mathbf{p} \in [0, 1]^n$ to $\partial\mathcal{S}$, then apply Prop. 4 in §III-B to further restrict it to $\partial\mathcal{S}_{1,2}$, and finally Thm. 1 is proved by employing Prop. 3 in §III-B and Prop. 6 below, the proof of which is found in Appendix II-A.

Leveraging the (p_s, k, n') parameterization of \mathbf{p} in Def. 1, recall the definition of $T(p_s, k, n')$ in (15) in §III and observe the Jain objective $F_{-1}(\mathbf{x}(\mathbf{p}))$ in (20) may be written as

$$F_{-1}(p_s, k, n') \equiv F_{-1}(\mathbf{x}(\mathbf{p}(p_s, k, n'))). \quad (22)$$

Prop. 6 establishes two key monotonicity properties of the objective (22) under the throughput equality constraint over the restricted set $\mathbf{p} \in \partial\mathcal{S}_2$.

Proposition 6: Under the constraints $\mathbf{p} \in \partial\mathcal{S}_2$ (with $\mathbf{p} = \mathbf{p}(p_s, k, n')$) and $T(p_s, k, n') = \theta$, the objective $F_{-1}(p_s, k, n')$ (22) obeys the following two monotonicity properties for all $(k, n') \in \mathcal{D}_{t,n}$ defined in (18):

- 1) $F_{-1}(p_s, k, n') < F_{-1}(p_s, k+1, n')$
- 2) $F_{-1}(p_s, k, n') < F_{-1}(p_s, k+1, n'+1)$.

In Fig. 1 (left), the two monotonicity results show F_{-1} is decreasing in k along any vertical line (fixed n'), and along any diagonal line with unit slope (fixed $n_t = n' - k$).

B. Main results

For general (n, θ) , where $n > 2$ and $\theta \in (0, 1)$, we are not able to obtain an *explicit* expression for the throughput–fairness tradeoff, primarily because there is no known explicit characterization of Λ for $n > 2$. If \mathbf{x}^* is an optimal

- 1) if $\theta < \theta_n$, then the maximum fairness is $F_J^* = 1$, achieved when every user receives equal rate: $x_i(\mathbf{p}^*) = \theta/n$.
- 2) if $\theta = \theta_t$ for some $t \in [n]$, then $\mathbf{p}^* = (1/t) \sum_{i=1}^t \mathbf{e}_i$, with the corresponding maximum fairness $F_J^* = t/n$.

$$\tilde{T}(F) = \left(1 - \frac{1}{nF}\right)^{nF-1} \quad (23)$$

3) if $\theta \in (\theta_t, \theta_{t-1})$ for some $t \in \{2, \dots, n\}$, then $\mathbf{p}^* = p_s^* \mathbf{e}_1 + p_l^* \sum_{i=2}^t \mathbf{e}_i$ where $p_l^* = p_l(p_s^*, k^*, n'^*)$ according to (12) with $k^* = 1$, $n'^* = t$, and p_s^* the unique real root on $(0, 1/t)$ of the following (order- t) polynomial (in p_s) equation:

The proof is found in Appendix II-B. The T-F tradeoff plots for $n = \{1, \dots, 4\}$ users are illustrated in Fig. 4 (right where regime 1) is omitted.

Remark 2: It can be verified that in the statement of Thm. 1, regime 2) can be merged into 3) by allowing (24) to be solved for p_s^* on $(0, 1/t]$. They are stated separately for conceptual clarity and better consistency with the proof of Thm. 2. In addition, regime 2) is where we have a closed-form expression for both the extremizer and the optimized objective.

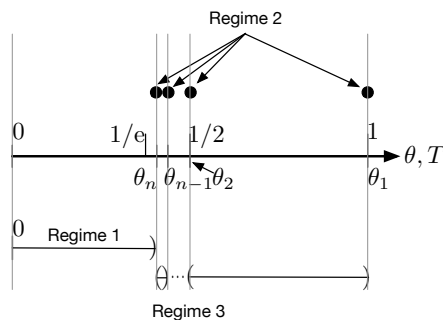


Fig. 3. Illustration of the three regimes, parameterized by θ , in Thm. 1: regime 1 is $\theta \in (0, \theta_n)$, regime 2 is $\theta = (\theta_1, \dots, \theta_n)$, and regime 3 is $\bigcup_{t=2}^n (\theta_t, \theta_{t-1})$.

As motivated in §II-D, the throughput inequality constraint is natural from the operational perspective of wishing to maximize fairness subject to a minimum throughput requirement. As may be intuitive, this modification to the

constraint (feasible set) has no effect on the solution, as shown in the following theorem.

Theorem 2: The solution in Thm. 1 of the Jain's throughput-fairness tradeoff (20) is unaffected by changing the throughput equality constraint to an inequality constraint $T(\mathbf{x}(\mathbf{p})) \geq \theta$.

The proof is found in Appendix II-B.

C. Properties of the Jain T-F tradeoff

As can be seen from Thm. 1, the extremizer $\mathbf{p}^* = \mathbf{p}(p_s^*, k^*, n'^*)$, with p_s^* solving $T(p_s, k^*, n'^*) = \theta$ in (24), has the property that n'^* , the total number of *active* users (i.e., users with nonzero contention probabilities), equals t , where $\theta \in [\theta_t, \theta_{t-1})$, for $t \in \{2, \dots, n\}$. In fact, because (24) does not depend on n , the total number of users in the system, one can easily verify that, if $\theta \geq \theta_{n-l}$ for some integer $l \in \{1, \dots, n-2\}$, then the extremizer \mathbf{p}^* is as if the total number of users in the system were $n-l$, except that l zeros need to be padded in order to make \mathbf{p}^* an n -dimensional vector. It follows that the maximum Jain's fairness satisfies

$$F_J^*(\theta; n) = \left(1 - \frac{l}{n}\right) F_J^*(\theta; n-l), \quad \theta \geq \theta_{n-l}, \quad (25)$$

where our notation highlights F_J^* is a function of θ and is parameterized by n .

One use of the recursive relationship (25) is that it enables incremental plotting of the T-F tradeoff for a sequence of values of $n \in \{2, \dots, n_{\max}\}$. From Thm. 1 if $\theta \in [\theta_n, \theta_{n-1})$ then $n'^* = n$, meaning, at the optimum, every user in the system is active. We therefore call the interval $[\theta_n, \theta_{n-1})$, for each $n \in \mathbb{N}$, the *active throughput interval*, meaning all n users are actively contending under the optimal control for any target throughput θ in this interval. This observation is the root idea in the Jain T-F plotting algorithm (Alg. 1), which returns a plot of the Jain T-F tradeoff over $\theta \in (0, 1)$ for all $n \in \{2, \dots, n_{\max}\}$. Naturally, the interval $[\theta_n, \theta_{n-1})$ must be discretized for each n . Fig. 4 (left) illustrates Alg. 1 for $n_{\max} = 4$ users. First, the plot of $F_J^*(\theta; 2)$ over $\theta \in [\theta_2, \theta_1)$ (i.e., the active interval for $n = 2$, thick blue) is scaled using (25) to obtain $F_J^*(\theta; 3)$ and $F_J^*(\theta; 4)$ over the same interval (thin blue for both). Then, the plot of $F_J^*(\theta; 3)$ over $\theta \in [\theta_3, \theta_2)$ (i.e., the active interval for $n = 3$, thick purple) is scaled to obtain $F_J^*(\theta; 4)$ over the same interval (thin purple), and so on. Note first that, for each n , at $\theta = 1$ the maximum Jain's fairness is the *minimum* possible, i.e., $F_J^* = 1/n$, corresponding to the fairness when only one user (say i) contends for access (i.e., $\mathbf{x} = \mathbf{p} = \mathbf{e}_i$), as $\mathbf{x} = \mathbf{e}_i$ is the unique (up to permutation) rate vector in Λ achieving $\theta = 1$. Second, for each n , for any $\theta \leq \theta_n$ the maximum Jain's fairness is the *maximum* possible, i.e., $F_J^* = 1$, corresponding to all n users contending with equal probability, uniquely achievable by the rate vector $\mathbf{x} = \theta \mathbf{u}$. The Jain T-F tradeoff for each n up to 4 users is shown in Fig. 4 (right).

The following theorem gives some properties of the optimal controls, optimal rates, and the Jain T-F tradeoff.

Theorem 3: The Jain T-F tradeoff for $n \geq 2$ users, over $\theta \in [\theta_n, 1)$, has the following properties:

- 1) For fixed n , the small and large contention probabilities of the optimal control, $p_s^*(\theta), p_l^*(\theta)$, and the corresponding optimal rates, $x_s^*(\theta), x_l^*(\theta)$, are piecewise decreasing and increasing, respectively, in θ . More precisely, fix $t \in \{2, \dots, n\}$ and $\theta \in [\theta_t, \theta_{t-1})$. Then:

Algorithm 1 Jain T-F tradeoff for all $n \in \{2, \dots, n_{\max}\}$

```

1: for  $n = 2, \dots, n_{\max}$  do
2:   Plot  $F_J^*(\theta; n) = 1$  for  $\theta \in [0, \theta_n)$ 
3:   for  $\theta \in [\theta_n, \theta_{n-1})$  do
4:     Compute  $p_s^*(\theta)$  solving  $T(p_s, 1, n) = \theta$  (i.e., (24) in Thm. 1 with  $t = n$ )
5:     Compute  $F_J^*(\theta; n) = F_J(\mathbf{x}(\mathbf{p}(p_s^*(\theta), 1, n)))$  using (2), (7), and Def. 1
6:   end for
7:   Plot  $F_J^*(\theta; m) = \frac{n}{m} F_J^*(\theta; n)$  for  $m \in \{n, \dots, n_{\max}\}$ 
8: end for

```

- a) Both p_s^* and x_s^* are continuous and decreasing over each interval $[\theta_t, \theta_{t-1})$, but are not monotone over $[\theta_n, 1)$. In particular, i) $\frac{dp_s^*(\theta)}{d\theta} < 0$, $\frac{dx_s^*(\theta)}{d\theta} < 0$, ii) at $\theta = \theta_t$ they take values $p_s^*(\theta_t) = 1/t$, $x_s^*(\theta_t) = \theta_t/t$, and iii) at $\theta = \theta_{t-1}$ they take value $p_s^*(\theta_{t-1}) = x_s^*(\theta_{t-1}) = 0$.
- b) Both p_l^* and x_l^* are continuous and increasing over $[\theta_n, 1)$, but neither is differentiable at each θ_t for $t \in \{2, \dots, n-1\}$. In particular, i) $\frac{dp_l^*(\theta)}{d\theta} > 0$, $\frac{dx_l^*(\theta)}{d\theta} > 0$, ii) at $\theta = \theta_t$ they take values $p_l^*(\theta_t) = 1/t$, $x_l^*(\theta_t) = \theta_t/t$, and iii) at $\theta = \theta_{t-1}$ they take value $p_l^*(\theta_{t-1}) = 1/(t-1)$ and $x_l^*(\theta_{t-1}) = \theta_{t-1}/(t-1)$.
- 2) For fixed n , the T-F tradeoff curve is decreasing in θ , i.e., $\frac{d}{d\theta} F_J^*(\theta; n) < 0$.
- 3) For fixed θ , the T-F tradeoff curve is decreasing in n , i.e., $F_J^*(\theta; n) > F_J^*(\theta; n+1)$.
- 4) For fixed n , the T-F tradeoff curve is continuous but nondifferentiable at $\{\theta_t\}_{t=2}^{n-1}$, i.e., $F_J^*(\theta; n)|_{\theta \downarrow \theta_t} = F_J^*(\theta; n)|_{\theta \uparrow \theta_t}$, but $\frac{d}{d\theta} F_J^*(\theta; n)|_{\theta \downarrow \theta_t} \neq \frac{d}{d\theta} F_J^*(\theta; n)|_{\theta \uparrow \theta_t}$ for each $t \in \{2, \dots, n-1\}$.
- 5) For fixed n , the T-F tradeoff curve is piecewise convex in θ , i.e., $\frac{d^2}{d\theta^2} F_J^*(\theta; n) > 0$, for $\theta \in [\theta_t, \theta_{t-1})$ with $t \in \{2, \dots, n\}$.

The proof is found in Appendix II-C. Fig. 5 shows $p_s^*(\theta)$, $p_l^*(\theta)$ (left) and $x_s^*(\theta)$, $x_l^*(\theta)$ (right), illustrating property 1) in Thm. 3. Properties 2) through 5) in Thm. 3 can be seen from Fig. 4 (right). Finally, we mention that a plot of the interpolated function $\tilde{T}(F)$ (23) in Thm. 1 (not shown) on the actual T-F tradeoff in Fig. 4 would show the interpolation lies above the true tradeoff, and is tight only at the critical throughputs θ .

V. α -FAIR NETWORK UTILITY MAXIMIZATION

In this section we investigate the throughput-fairness tradeoff within the framework of α -fair utility functions [6], [7]. Recall the objective F_α (for $\alpha \geq 0$), the α -fair utility function U_α , the throughput function T , and the mapping between a control \mathbf{p} and a rate vector $\mathbf{x}(\mathbf{p})$ given in (8), (9), (5), and (2) respectively. The optimization under a throughput equality constraint is:

$$\max_{\mathbf{p} \in [0,1]^n} F_\alpha(\mathbf{x}(\mathbf{p})) = \sum_{i=1}^n U_\alpha(x_i(\mathbf{p})) \text{ s.t. } T(\mathbf{x}(\mathbf{p})) = \theta. \quad (26)$$

We solve this problem for $\alpha \geq 1$. In this following we give i) preliminary results in §V-A, ii) the main results in §V-B, and iii) some additional properties of the α -fair throughput-fairness tradeoff in §V-C.

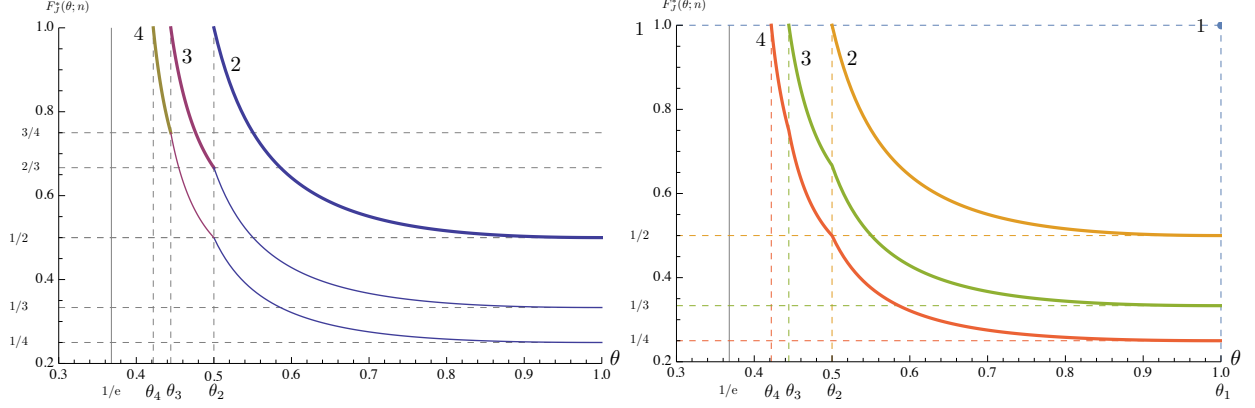


Fig. 4. **Left:** Illustration of using Alg. 1, leveraging the Jain fairness recursion (25), to incrementally plot the Jain T-F tradeoff for $n_{\max} = 4$ users. Vertical gridlines indicate the θ_t 's for $t \in \{2, 3, 4\}$. Horizontal gridlines indicate the maximum fairness at the θ_t 's for each $t \in [n]$ and each $n \in n_{\max}$. The T-F tradeoff for the active throughput intervals (thick curves) need to be computed first, after which the rest parts (thin curves) can be obtained by scaling. **Right:** Thm. 1 (regimes 2) and 3)): T-F tradeoff under Jain's fairness for $n = 1$ (blue), 2 (orange), 3 (green), and 4 (red).

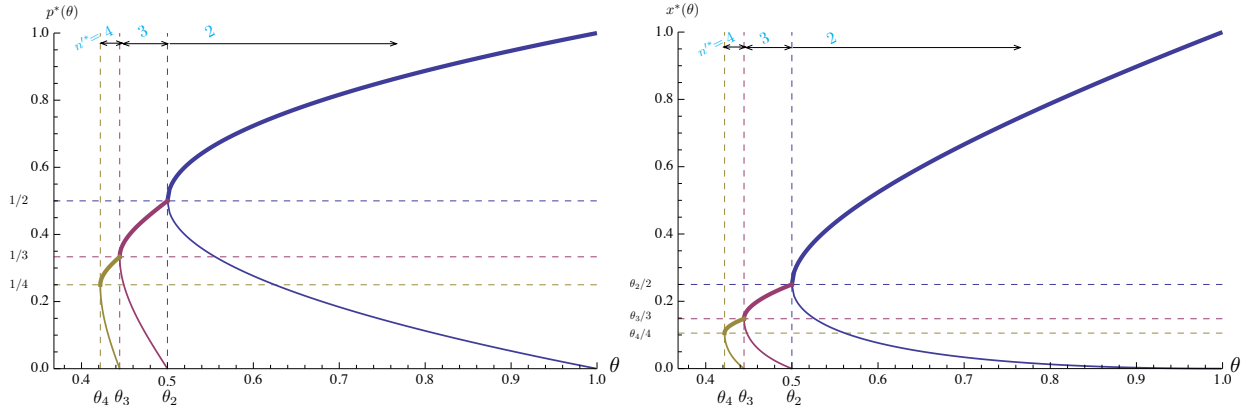


Fig. 5. Illustration of property 1) in Thm. 3: Optimal controls $p_s^*(\theta)$ (left, lower/thinner branches), $p_l^*(\theta)$ (left, upper/thicker branches) and optimal rates $x_s^*(\theta)$ (right, lower/thinner branches), $x_l^*(\theta)$ (right, upper/thicker branches) versus target throughput θ , for $n = 4$ users. Vertical gridlines indicate the θ_t 's: $(\theta_2, \theta_3, \theta_4) = (\frac{1}{2}, \frac{4}{9}, \frac{27}{64}) \approx (0.5, 0.4444, 0.4219)$. Horizontal gridlines indicate the corresponding optimal controls (left) and optimal rates (right) when $\theta = \theta_t$. Shown also are the optimal number of active users n^{t*} for different ranges of θ .

A. Preliminary results

We start with the special case $n = 2$.

Proposition 7: The throughput–fairness tradeoff under α -fairness ($\alpha \geq 1$), for $n = 2$ users, is

$$F_\alpha^*(\theta) = \begin{cases} -\frac{2}{\alpha-1} \left(\frac{2}{\theta}\right)^{\alpha-1}, & \theta \in (0, \frac{1}{2}] \quad \alpha > 1 \\ -\frac{1}{\alpha-1} \left(\left(\frac{\theta+\sqrt{2\theta-1}}{2}\right)^{1-\alpha} + \left(\frac{\theta-\sqrt{2\theta-1}}{2}\right)^{1-\alpha} \right), & \theta \in (\frac{1}{2}, 1) \\ -2 \log \frac{2}{\theta}, & \theta \in (0, \frac{1}{2}] \quad \alpha = 1 \\ -2 \log \frac{2}{1-\theta}, & \theta \in (\frac{1}{2}, 1) \end{cases} \quad (27)$$

Proof: The proof resembles that of Prop. 5 in §IV-A. The all-rates equal ray $\{\mathbf{x} : x_1 = x_2\}$ can still be viewed as the maximum fairness line as the maximum α -fair objective is attained by points either on this line or closest to this line, subject to the throughput constraint $x_1 + x_2 = \theta$. This follows from the Schur-concavity of the objective (Prop. 1 in §III-A) and (the proof of) Cor. 1 in §III-A. Therefore, when $\theta \leq 1/2$, the maximizer is on the ray $\{\mathbf{x} : x_1 = x_2\}$ and hence $(x_1^*, x_2^*) = (\frac{\theta}{2}, \frac{\theta}{2})$; when $\theta > 1/2$, the maximizer is obtained by finding the points on the boundary of Λ that satisfy the throughput constraint (as they are the closest to the all-rates equal ray, see Fig. 8), which gives $(x_1^*, x_2^*) = \left(\frac{\theta \pm \sqrt{2\theta-1}}{2}, \frac{\theta \mp \sqrt{2\theta-1}}{2}\right)$. Substitution of the expressions of the maximizers into the objective yields (27). ■

The basic idea in solving the throughput-fairness tradeoff under α -fairness (Thm. 4) is to first apply Cor. 1 in §III-A to restrict the feasible set from $\mathbf{p} \in [0, 1]^n$ to ∂S , and then apply Prop. 4 in §III-B to further restrict it to $\partial S_{1,2}$. The optimization problem is solved with the aid of Prop. 8 shown below, which establishes a key monotonicity property of the objective in (26) under the throughput equality constraint over the restricted set $\mathbf{p} \in \partial S_2$. It plays a similar role to that of Prop. 6 in proving Thm. 1 (§IV-B).

Leveraging the (p_s, k, n') parameterization of \mathbf{p} in Def. 1 and the definition of $T(p_s, k, n')$ in (15) in §III-B we define

$$F_\alpha(p_s, k, n') \equiv F_\alpha(\mathbf{x}(\mathbf{p}(p_s, k, n'))). \quad (28)$$

Proposition 8: Under the constraints $\mathbf{p} \in \partial S_2$ (with $\mathbf{p} = \mathbf{p}(p_s, k, n')$) and $T(p_s, k, n') = \theta$, the objective $F_\alpha(p_s, k, n')$ (28) for $\alpha \geq 1$ is increasing in k for $k \in \{1, \dots, n' - 1\}$ when n' is held fixed. Thus the maximum of $F_\alpha(p_s, k, n')$ is attained when $k^* = n' - 1$.

The proof is found in Appendix III-A.

B. Main results

For general (n, θ) , where $n > 2$ and $\theta \in (0, 1)$, we will again give an *implicit* characterization of the T-F tradeoff under α -fairness when $\alpha \geq 1$. The main theorem of this subsection is a characterization of the optimal control \mathbf{p}^* for each θ (as the solution of a polynomial equation) from which we can compute $F_\alpha(\mathbf{x}(\mathbf{p}^*))$.

Theorem 4 (Throughput-fairness tradeoff under α -fair when $\alpha \geq 1$): The throughput-fairness tradeoff for $n \geq 2$ users under α -fairness when $\alpha \geq 1$, with a throughput equality constraint $T(\mathbf{x}) = \theta$, for $\theta \in (0, 1)$, includes two regimes, parameterized by θ :

- 1) if $\theta \leq \theta_n$, then the maximum fairness is

$$F_\alpha^*(\theta) = \begin{cases} -n \log\left(\frac{n}{\theta}\right), & \alpha = 1 \\ -\frac{n}{\alpha-1} \left(\frac{n}{\theta}\right)^{\alpha-1}, & \alpha > 1 \end{cases}, \quad (29)$$

achieved when every user receives equal rate: $x_i(\mathbf{p}^*) = \theta/n$.

- 2) if $\theta \in (\theta_n, 1)$, then $\mathbf{p}^* = p_s^* \mathbf{e}_1 + p_l^* \sum_{i=2}^n \mathbf{e}_i$ where $p_l^* = p_l(p_s^*, k^*, n'^*)$ according to (12) with $k^* = n - 1$, $n'^* = n$, and p_s^* the unique real root on $(0, 1/n)$ of the following polynomial equation

$$((n-1)p_s)^2 (1-p_s)^{n-2} + (1-(n-1)p_s)(1-p_s)^{n-1} = \theta. \quad (30)$$

The proof is found in Appendix III-B. The T-F tradeoff plots for $n = \{1, \dots, 4\}$ users are illustrated in Fig. 6.

Observe the difference between regime 1) in Thm. 4 for α -fairness when $\alpha \geq 1$ and regime 1 in Thm. 1 for Jain's fairness: although the maximizers are the same, the objective is increasing in θ in the former, whereas it is constant in the latter. Observe also the asymmetry between regime 2) in Thm. 4 and regimes 2) and 3) in Thm. 1: $k^* = n'^* - 1$ and $n'^* = n$ for all $\theta \in (\theta_n, 1)$ in the former, while $k^* = 1$ and $n'^* = t$ for $\theta \in [\theta_t, \theta_{t-1})$ in the latter. Thus, the optimal control vector \mathbf{p}^* for α -fairness has $n'^* - 1$ users with "small" contention probability p_s^* and one user with "large" contention probability p_l^* for n'^* always equal to n , while the optimal control vector \mathbf{p}^* for Jain's fairness has one user with p_s^* and $n'^* - 1$ users with p_l^* , for n'^* determined by the active throughput interval containing θ .

Similar to §IV-B, we now address the case where the throughput constraint in (26) is an inequality $T(\mathbf{x}(\mathbf{p})) \geq \theta$.

Theorem 5: If the throughput equality constraint is changed to an inequality constraint $T(\mathbf{x}(\mathbf{p})) \geq \theta$ then the solution in Thm. 4 of the α -fair utility maximization problem (26) when $\alpha \geq 1$ is only affected in the first regime, namely when $\theta \leq \theta_n$. More precisely, if $\theta \leq \theta_n$, then the maximum fairness is independent of θ and is given by

$$F_\alpha^*(\theta) = \begin{cases} -n \log\left(\frac{n}{\theta_n}\right), & \alpha = 1 \\ -\frac{n}{\alpha-1} \left(\frac{n}{\theta_n}\right)^{\alpha-1}, & \alpha > 1 \end{cases}, \quad (31)$$

where the maximizer in the control space is a uniform vector $\mathbf{p}^* = \mathbf{u}$.

The proof is found in Appendix III-B.

C. Properties of the α -fair T-F tradeoff

The follow theorem gives some properties of the T-F tradeoff for the α -fair objective.

Theorem 6: The T-F tradeoff for $n \geq 2$ users under α -fairness for $\alpha \geq 1$, with target throughput $\theta \in (\theta_n, 1)$, has the following properties:

- 1) For fixed α and n , the smaller (p_s^*) and larger (p_l^*) components of the optimal control are decreasing and increasing in θ respectively, i.e., $\frac{dp_s^*(\theta)}{d\theta} < 0$, $\frac{dp_l^*(\theta)}{d\theta} > 0$. The smaller (x_s^*) and larger (x_l^*) components of the corresponding optimal rate vectors are likewise decreasing and increasing in θ , i.e., $\frac{dx_s^*(\theta)}{d\theta} < 0$, $\frac{dx_l^*(\theta)}{d\theta} > 0$.
- 2) For fixed α and n , the maximum α -fair objective (F_α^*) is decreasing in θ i.e., $\frac{d}{d\theta} F_\alpha^*(\theta; n) < 0$, and is continuous and differentiable. For $n = 2$, $F_\alpha^*(\theta; 2)$ is concave (i.e., $\frac{d^2}{d\theta^2} F_\alpha^*(\theta; 2) < 0$). For $n > 2$, there exists a throughput threshold $\hat{\theta}_\alpha(n)$ such that $F_\alpha^*(\theta; n)$ is convex (concave) in θ for $\theta < \hat{\theta}_\alpha(n)$ ($\theta > \hat{\theta}_\alpha(n)$).
- 3) For fixed α and $\theta \in (\theta_n, 1)$, the maximum α -fair objective is decreasing in n , i.e., $F_\alpha^*(\theta; n) > F_\alpha^*(\theta; n+1)$.

The proof is found in Appendix III-C. Fig. 7 shows $p_s^*(\theta), p_l^*(\theta)$ (left) and $x_s^*(\theta), x_l^*(\theta)$ (right), illustrating property 1) in Thm. 6. Fig. 6 illustrates properties 2) and 3) for the cases of $\alpha = 1$ (left) and $\alpha = 2$ (right).

VI. CONCLUSION

We have presented six theorems that characterize the throughput-fairness tradeoff under slotted Aloha, using both Jain's fairness measure (Theorems 1-3), and the α -fair measure (Theorems 4-6). The key property enabling

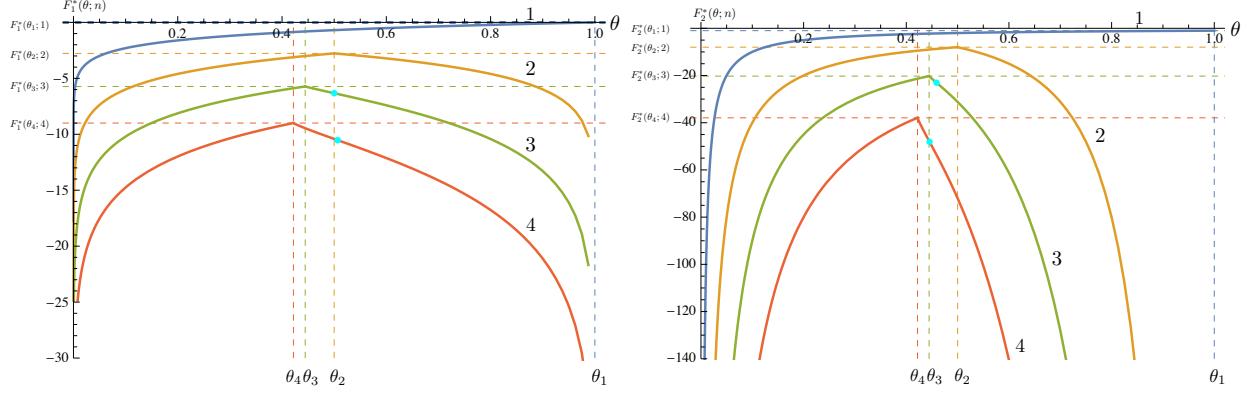


Fig. 6. Illustration of Thm. 4 and properties 2) and 3) in Thm. 6: T-F tradeoff under α -fairness when $n = 1$ (blue), 2 (orange), 3 (green), and 4 (red) users, for $\alpha = 1$ (left) and $\alpha = 2$ (right). Vertical gridlines indicate the θ_t 's and horizontal gridlines indicate the corresponding optimal α -fair objective for each n at $\theta = \theta_n$ i.e., $F_\alpha^*(\theta_n; n)$. Shown as cyan dots are the “inflection” points upon which the T-F curves transitions from convex decreasing to concave decreasing, for $n > 2$. The thresholding $\hat{\theta}_\alpha(n)$ is computed using (143).

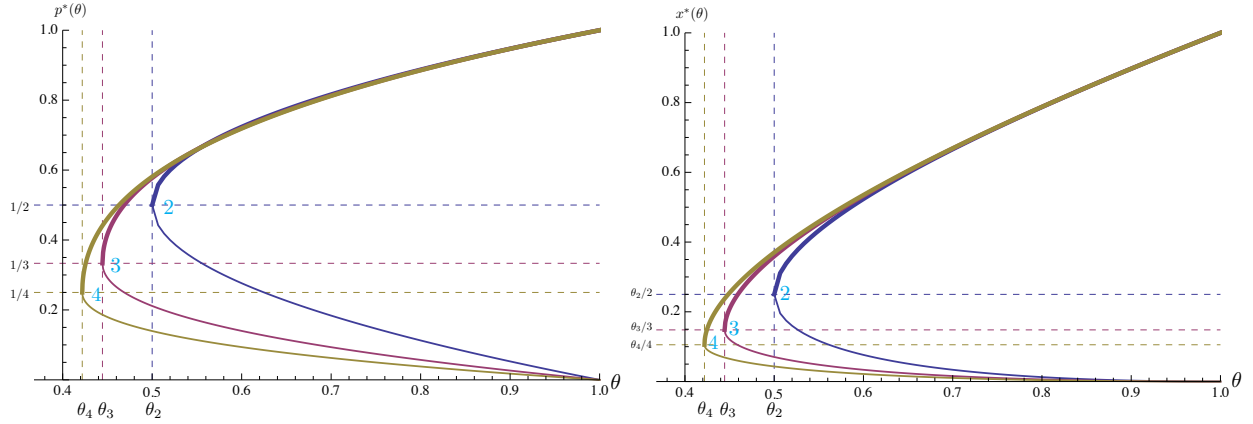


Fig. 7. Illustration of property 1) in Thm. 6: Optimal controls $p_s^*(\theta)$ (left, lower/thinner branches), $p_t^*(\theta)$ (left, upper/thicker branches) and optimal rates $x_s^*(\theta)$ (right, lower/thinner branches), $x_t^*(\theta)$ (right, upper/thicker branches) versus target throughput θ , for $n = 2$ (blue), 3 (purple), and 4 (yellow) users. Vertical gridlines indicate the θ_t 's: $(\theta_2, \theta_3, \theta_4) = (\frac{1}{2}, \frac{4}{9}, \frac{27}{64}) \approx (0.5, 0.4444, 0.4219)$. Horizontal gridlines indicate the corresponding optimal controls (left) and optimal rates (right) when $\theta = \theta_t$. Different from the case of Jain's fairness (see Fig. 5 in §IV-C where only the plots for $n = 4$ users are shown), here $n'^* = n$ holds irrespective of the value of θ .

the analysis is Prop. 4, which reduces the set of potential extremizers of the fairness functions from $[0, 1]^n$ to $\partial S_{1,2}$, i.e., those controls taking at most two nonzero values. Theorems 1 and 3 address the case of a throughput equality constraint, $T(\mathbf{x}) = \theta$, and Theorems 2 and 4 address the case of a throughput inequality constraint $T(\mathbf{x}) \geq \theta$. The main point is that the throughput–fairness tradeoff is the same for both types of constraints (for $\theta \geq \theta_n$). The key difference between the Jain and α -fair tradeoff under a throughput constraint $\theta \in [\theta_t, \theta_{t-1})$ is in the nature of the optimal controls: to maximize the Jain fairness objective requires $n'^* = t$ active users, of which $k^* = 1$ use a small contention probability and rate and $t - 1$ use a large contention probability and rate, while to maximize the α -fair

objective requires *all* ($n'^* = n$) users be active with $k^* = n - 1$ small users, and one large user. Perhaps the most surprising result (to us) is the fact that the Jain throughput-fairness tradeoff is piecewise convex over each critical throughput interval $[\theta_t, \theta_{t-1})$ for $t \in [n]$, but not convex overall, i.e., over $[\theta_n, 1)$.

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APPENDIX I

PROOFS FROM §III

Proofs from §III-A and §III-B are given in Appendix I-A and Appendix I-B respectively.

A. Proofs from §III-A

The following lemma is used in the proof of Prop. 2, below.

Lemma 2: Fix a set of $m \geq 2$ points $\mathcal{V} \equiv \{v_1, \dots, v_m\} \subset \mathbb{R}^n$ such that no v_i can be expressed as a convex combination of any other points in \mathcal{V} , and denote by $\mathcal{C}_h \equiv \text{conv}(\mathcal{V})$ the *convex hull* of \mathcal{V} . Fix a *strictly convex* set, denoted \mathcal{C}_s , whose boundary also includes the set \mathcal{V} , namely $\partial\mathcal{C}_s \supseteq \mathcal{V}$. Then the boundary of \mathcal{C}_s intersects \mathcal{C}_h only at the m points that generate \mathcal{C}_h , namely $\partial\mathcal{C}_s \cap \mathcal{C}_h = \mathcal{V}$.

Proof: Note $\mathcal{V} \subseteq \partial\mathcal{C}_s \cap \mathcal{C}_h$ by assumption. We need to show the intersection $\partial\mathcal{C}_s \cap \mathcal{C}_h$ can never include any other point. Recall a set \mathcal{A} is strictly convex if for any $x, y \in \mathcal{A}$, every point on the *line segment* connecting x and y other than the end points is in the interior of \mathcal{A} . First we observe $\mathcal{C}_s \supseteq \mathcal{C}_h$, by virtue of the fact that the convex hull is the smallest convex set that contains \mathcal{V} . Second, we prove by contradiction that the intersection $\partial\mathcal{C}_s \cap \mathcal{C}_h$ can only consist of points on the boundary of \mathcal{C}_h (denoted $\partial\mathcal{C}_h$). Assume there exists a point $v_{\text{int}} \in \partial\mathcal{C}_s \cap \mathcal{C}_h$ that is an interior point of \mathcal{C}_h . This means there exists a neighborhood of v_{int} that resides in \mathcal{C}_h , however, as v_{int} is also on the boundary of \mathcal{C}_s , every neighborhood of v_{int} must contain points that belong to neither \mathcal{C}_s nor \mathcal{C}_h (as $\mathcal{C}_s \supseteq \mathcal{C}_h$). This contradiction shows $\partial\mathcal{C}_s \cap \mathcal{C}_h \subseteq \partial\mathcal{C}_h$. Observe that, since \mathcal{C}_h is a polytope, it has the property that any point on its boundary aside from the vertices, i.e., $v \in \partial\mathcal{C}_h \setminus \mathcal{V}$, may be expressed as a strict convex combination of two other points on the boundary, say $v', v'' \in \partial\mathcal{C}_h$. Third, the previous sentence applies to any point $v \in (\partial\mathcal{C}_s \cap \mathcal{C}_h) \setminus \mathcal{V}$, since such points are in $\partial\mathcal{C}_h \setminus \mathcal{V}$. But the implied ability to represent v as a strict convex combination of $v', v'' \in \mathcal{C}_s$ violates the assumed strict convexity of \mathcal{C}_s , since it implies a boundary point of \mathcal{C}_s lies on the open line segment formed by two other points in \mathcal{C}_s . This establishes no such point exists, thereby proving the lemma. ■

Proof of Prop. 2: Write $\Pi(\mathbf{x})$ to denote the $n!$ permutations of \mathbf{x} . For item 1), we apply Prop. C.1 in Ch. 4 of [26, pp. 162] (Rado, 1952) which says $\mathbf{a} \prec \mathbf{b}$ if and only if \mathbf{a} lies in the convex hull of the $n!$ permutations of \mathbf{b} , denoted $\text{conv}(\Pi(\mathbf{b}))$. Let $\mathbf{x} \in \Lambda_\theta^{\text{int}}$; it suffices to establish $\mathbf{x}' \in \partial\Lambda_\theta$ with $\mathbf{x}' \prec \mathbf{x}$. The geometric argument

below is illustrated in Fig. 8 by replacing \mathbf{x}^* in the figure with \mathbf{x}' . Define $\mathbf{c} \equiv \theta \mathbf{u}$. First: it follows from Lem. 1 that $\mathbf{c} \notin \Lambda$ (since $\theta > \theta_n$), but that $\mathbf{c} \in \text{conv}(\Pi(\mathbf{x}))$ (using the convex combination of $\Pi(\mathbf{x})$ with all weights equal to $1/n!$). Second: it follows from the convexity⁵ of Λ^c that there exists a unique point $\mathbf{x}' \in \partial\Lambda$ on the line segment connecting \mathbf{x} with \mathbf{c} . Third: it follows from the convexity of \mathcal{H}_θ that $\mathbf{x}' \in \mathcal{H}_\theta$ (which contains both \mathbf{x}, \mathbf{c}), and therefore, $\mathbf{x}' \in \partial\Lambda_\theta$ (as it lies in both $\partial\Lambda$ and \mathcal{H}_θ). Fourth: this point $\mathbf{x}' \in \text{conv}(\Pi(\mathbf{x}))$ by the convexity of $\text{conv}(\Pi(\mathbf{x}))$ (which contains both \mathbf{x}, \mathbf{c}). Fifth: by Rado's result, $\mathbf{x}' \prec \mathbf{x}$, which concludes the proof of item 1).

For item 2), we again apply Rado's result and prove by contradiction. Assume there exist distinct (up to permutation) \mathbf{x}, \mathbf{x}' both in $\partial\Lambda_\theta$ satisfying $\mathbf{x}' \prec \mathbf{x}$, equivalently, $\mathbf{x}' \in \text{conv}(\Pi(\mathbf{x})) \equiv \mathcal{C}_h$. The contradiction will establish $\partial\Lambda_\theta \cap \mathcal{C}_h = \Pi(\mathbf{x})$, meaning the only feasible points (i.e., in $\partial\Lambda_\theta$) that are majorized by \mathbf{x} (i.e., in \mathcal{C}_h) are permutations of the original point \mathbf{x} . This provides the desired contradiction since permutations of a point do not majorize each other. Our approach to establishing $\partial\Lambda_\theta \cap \mathcal{C}_h = \Pi(\mathbf{x})$ is to apply Lem. 2, with $\mathcal{V} = \Pi(\mathbf{x})$ and $\mathcal{C}_s = \Lambda_\theta^c = \Lambda^c \cap \mathcal{H}_\theta$. To apply Lem. 2 we must show *i)* \mathcal{C}_s is strictly convex, and *ii)* $\partial\mathcal{C}_s \supseteq \mathcal{V}$, i.e., $\partial\Lambda_\theta \supseteq \Pi(\mathbf{x})$ (since $\partial\Lambda^c = \partial\Lambda$). The lemma establishes the desired result, $\partial\mathcal{C}_s \cap \mathcal{C}_h = \Pi(\mathbf{x})$. It remains to show *i)* and *ii)*. *i)* Subramanian and Leith [27, Lem. 1 and Remark 1 in §II-A] have shown that Λ^c is strictly convex⁶ in \mathbb{R}_+^n . As strict convexity is preserved under intersection with affine spaces, it follows that \mathcal{C}_s is strictly convex. *ii)* By assumption $\mathbf{x} \in \partial\Lambda_\theta$, which ensures $\Pi(\mathbf{x}) \subset \partial\Lambda_\theta$ since Λ and \mathcal{H}_θ are permutation invariant. This establishes item 2). ■

Proof of Cor. 1 for the case of Jain's fairness (7) (formulated as (20)):

Given that \mathbf{x}^* satisfies the throughput constraint $\mathbf{x}^* \in \mathcal{H}_\theta$, we need to show $\mathbf{x}^* \in \partial\Lambda$, i.e., the optimal rate vector \mathbf{x}^* is Pareto efficient. Refer to Fig. 8 for geometric intuition. Recall $\mathbf{0}$ denotes the origin and $\mathbf{m} \equiv \frac{1}{n}\theta_n \mathbf{1} \in \partial\Lambda$. Define the following: *i)* $\mathbf{c} = c\mathbf{1}$ with $c \equiv \theta/n$, *ii)* $\text{ray}(\mathbf{0}, \mathbf{1})$ as the ray emanating from $\mathbf{0}$ in the direction $\mathbf{1}$ (holding $\mathbf{0}, \mathbf{m}$, and \mathbf{c}). Recall *i)* $\mathcal{H}_\theta = \{\mathbf{x} : \sum_i x_i = \theta\}$ is the hyperplane with normal $\mathbf{1}$ (and thereby orthogonal to $\text{ray}(\mathbf{0}, \mathbf{1})$), and *ii)* $\mathbf{x} = \mathbf{x}(\mathbf{p}) \in \mathcal{H}_\theta \cap \Lambda$ is a feasible rate vector under the throughput constraint for feasible control \mathbf{p} . Observe \mathcal{H}_θ intersects with $\text{ray}(\mathbf{0}, \mathbf{1})$ at \mathbf{c} . Finally, note that the objective in (20) is $\frac{1}{2}d(\mathbf{x}, \mathbf{0})^2$.

Since \mathcal{H}_θ is orthogonal to $\text{ray}(\mathbf{0}, \mathbf{1})$, it follows that $(\mathbf{0}, \mathbf{c}, \mathbf{x})$ form a right triangle with the right angle at \mathbf{c} , and therefore, by the Pythagorean theorem, $d(\mathbf{x}, \mathbf{0})^2 = d(\mathbf{x}, \mathbf{c})^2 + d(\mathbf{c}, \mathbf{0})^2$. It follows that the objective $\frac{1}{2}d(\mathbf{x}, \mathbf{0})^2$ is minimized iff $d(\mathbf{x}, \mathbf{c})^2$ is minimized (over $\mathbf{x} \in \mathcal{H}_\theta \cap \Lambda$). Observe the assumption $\theta \geq \theta_n$ ensures $\mathbf{c} \notin \partial\Lambda$ for $\theta > \theta_n$, and $\mathbf{c} = \mathbf{m} \in \partial\Lambda$ for $\theta = \theta_n$ (in which case the unique global minimizer is $\mathbf{x}^* = \mathbf{m}$). Fix a candidate feasible point $\mathbf{x} \in \mathcal{H}_\theta \cap \Lambda$ and consider the line segment connecting \mathbf{x} with \mathbf{c} : it must intersect $\partial\Lambda$, and this point is denoted $\mathbf{x}^*(\mathbf{x})$. It is clear that any feasible \mathbf{x}' on the line segment (\mathbf{x}, \mathbf{c}) not equal to $\mathbf{x}^*(\mathbf{x})$ is suboptimal to $\mathbf{x}^*(\mathbf{x})$ in that $d(\mathbf{x}', \mathbf{c}) > d(\mathbf{x}^*(\mathbf{x}), \mathbf{c})$. This shows the desired minimizer $\mathbf{x}^* \in \partial\Lambda$. Equivalently ([25], recall (3)) this means the corresponding optimal control \mathbf{p}^* (in the sense of (2)) is in $\partial\mathcal{S}$. ■

⁵The complement of Λ , i.e., $\Lambda^c \equiv \mathbb{R}_+^n \setminus \Lambda$ is shown to be convex by Post in [25].

⁶Post [25] establishes the tangent hyperplane equation of every point on $\partial\Lambda$.

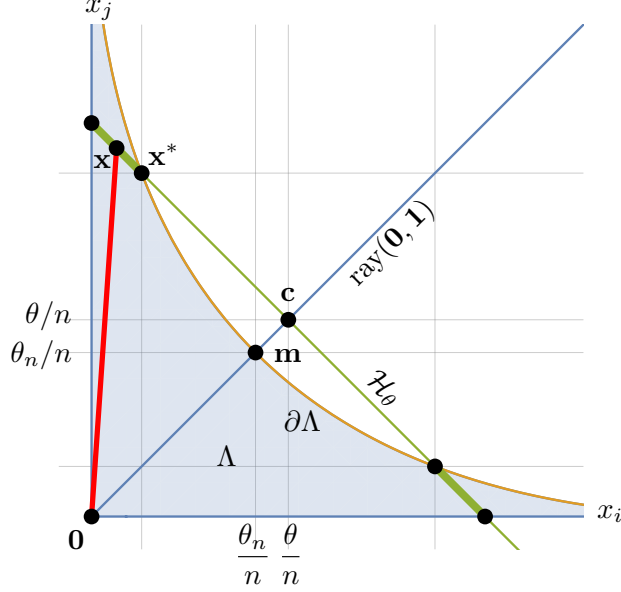


Fig. 8. Illustration of the proof of Cor. 1 for the case of Jain's fairness: $\mathbf{x}(\mathbf{p}^*) \in \partial\Lambda$. The feasible set under a throughput constraint $T(\mathbf{x}) = \theta$ is the intersection of Λ with the throughput hyperplane $\mathcal{H}_\theta = \{\mathbf{x} : \sum_i x_i = \theta\}$, indicated by the two bold green line segments.

B. Proofs from §III-B

Proof of Prop. 3: We prove the three statements in the order they are given.

Proof of 1). Observe by definition of $\mathbf{p} \in \overline{\partial\mathcal{S}_2}$ and $T(\mathbf{x}(\mathbf{p}))$, we may write $T(p_s, k, n') = kx_s + (n' - k)x_l$. Substituting the expressions for x_s, x_l in (13) in Def. 1 yields:

$$T(p_s, k, n') = -\frac{(1 - p_s)^{k-1} (-kn'p_s^2 + n'p_s + k - n)(1 - p_l)^{n'-k}}{kp_s + n' - k - 1}. \quad (32)$$

The partial derivative w.r.t. p_s is

$$\frac{\partial}{\partial p_s} T(p_s, k, n') = -\frac{k(1 - p_s)^{k-2}(n'p_s - 1)(1 - p_l)^{n'-k}(kp_s(n'p_s - 2) - (n' - 1 - k))}{(kp_s + n' - k - 1)^2}. \quad (33)$$

One can easily verify this derivative is nonpositive on the regime of interest, and thus $T(p_s, k, n')$ is monotone decreasing in p_s on $(0, 1/n']$, and as such there can exist at most one value of p_s solving $T(p_s, k, n') = \theta$.

Proof of 2). Since n' is fixed, we write $T(p_s, k, n')$ defined in (15) as $T(p_s, k)$. Observe that the monotonicity of $T(p_s, k)$ ensures $\mathcal{R}(k, n') = [T(1/n', k, n'), T(0, k, n')]$. Observe from (12) that $p_l(1/n', k, n') = p_s$ and $p_l(0, k, n') = 1/(n' - k)$. Substitution of $(p_s, p_l) = (1/n', 1/n')$ and $(p_s, p_l) = (0, 1/(n' - k))$ in (32) yields $\mathcal{R}(k, n') = [\theta_{n'}, \theta_{n'-k}]$. As $\theta_{n'}$ is constant in k , while $\theta_{n'-k}$ is increasing in k , it follows that the intervals forming each $\mathcal{R}(k, n')$ are nested and increasing in k .

Proof of 3). Recall i) $\theta_n \leq \dots \leq \theta_1$ (6), ii) $\mathcal{R}(k, n') = [\theta_{n'}, \theta_{n'-k}]$ (Item 2)), and iii) by assumption, the target θ lies in $[\theta_t, \theta_{t-1})$, for some $t \in \{2, \dots, n\}$. First, observe $n' \geq t$ needs to hold, since for $n' \leq t - 1$ we have

$$\mathcal{R}(k, n') \cap [\theta_t, \theta_{t-1}) = [\theta_{n'}, \theta_{n'-k}] \cap [\theta_t, \theta_{t-1}) = \emptyset. \quad (34)$$

Second, refer to Fig. 1 (right). As evident from the figure, $\theta \in \mathcal{R}(k, n')$ if and only if $k \in \{n' - t + 1, \dots, n' - 1\}$. ■

Proof of Prop. 4:

We prove the two statements in the order they are given.

Proof of i). The main idea of the proof is to establish the impossibility of any $\mathbf{p} \in [0, 1]^n$ simultaneously being an extremizer and having $|\mathcal{V}(\mathbf{p})| > 2$. Observe we may partition the feasible set $[0, 1]^n$ into $\{\mathbf{p} \in [0, 1]^n : |\mathcal{V}(\mathbf{p})| \leq 2\}$ and $\{\mathbf{p} \in [0, 1]^n : |\mathcal{V}(\mathbf{p})| > 2\}$. We now show any \mathbf{p} with $|\mathcal{V}(\mathbf{p})| > 2$ cannot satisfy the KKT conditions, given below, necessary for \mathbf{p} to be an extremizer.

We first consider the case of a throughput inequality constraint, $T(\mathbf{x}(\mathbf{p})) \geq \theta$. Introducing Lagrange multipliers μ_θ for $T(\mathbf{x}(\mathbf{p})) \geq \theta$, $\boldsymbol{\lambda} = (\lambda_i, i \in [n])$ for $\mathbf{p} \geq \mathbf{0}$, and $\boldsymbol{\nu} = (\nu_i, i \in [n])$ for $\mathbf{p} < \mathbf{1}$, the Lagrangian is:

$$\mathcal{L}(\mathbf{p}, \mu_\theta, \boldsymbol{\lambda}, \boldsymbol{\nu}) = F_\alpha(\mathbf{x}(\mathbf{p})) + \mu_\theta(\theta - T(\mathbf{x}(\mathbf{p}))) + \sum_{i=1}^n \lambda_i (-p_i) + \sum_{i=1}^n \nu_i (p_i - 1). \quad (35)$$

The first-order Karush-Kuhn-Tucker (KKT) necessary conditions for a *maximizer* are, for each $i \in [n]$:

$$\begin{aligned} \text{stationarity} \quad & \frac{\partial \mathcal{L}}{\partial p_i} = 0 \\ \text{primal feasibility} \quad & \theta - T(\mathbf{x}(\mathbf{p})) \leq 0 \\ & -p_i \leq 0, \quad p_i - 1 < 0 \\ \text{dual feasibility} \quad & \mu_\theta \leq 0, \lambda_i \leq 0, \nu_i \leq 0 \\ \text{comp. slackness} \quad & \mu_\theta(\theta - T(\mathbf{x}(\mathbf{p}))) = 0 \\ & \lambda_i (-p_i) = 0, \quad \nu_i (p_i - 1) = 0. \end{aligned}$$

The KKT conditions for a *minimizer* are the same, with the signs on each Lagrange multiplier on each inequality constraint reversed. As is evident from the proof below, the sign of the multipliers is inessential to establishing the result, and therefore the result holds for *both* minimization and maximization.

The first step of the proof is to derive the condition $g_k = 0$ in (40) below from the KKT stationarity condition $\frac{\partial \mathcal{L}}{\partial p_k} = 0$ when $0 < p_k < 1$. Towards that goal, we make the following definitions, where the dependence of these quantities upon \mathbf{p} is omitted for brevity:

$$\begin{aligned} F_\alpha &\equiv F_\alpha(\mathbf{x}(\mathbf{p})), & T &\equiv T(\mathbf{x}(\mathbf{p})) \\ \pi &= \pi(\mathbf{p}) \equiv \prod_j (1 - p_j), & \pi_i &= \pi_i(\mathbf{p}) \equiv \frac{\pi}{1 - p_i}. \end{aligned} \quad (36)$$

Observe $x_i = x_i(\mathbf{p})$ in (2) may be written in terms of π as $x_i = \frac{p_i}{1 - p_i} \pi$. Differentiation of (35) yields:

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial F_\alpha}{\partial p_i} - \mu_\theta \frac{\partial T}{\partial p_i} - \lambda_i + \nu_i. \quad (37)$$

The following partial derivatives may be established after some algebra:

$$\begin{aligned}\frac{\partial x_j(\mathbf{p})}{\partial p_i} &= \begin{cases} \frac{\pi}{1-p_j}, & i = j \\ -\frac{\pi p_j}{(1-p_i)(1-p_j)}, & i \neq j \end{cases} \\ \frac{\partial T}{\partial p_i} &= \frac{\pi_i - T}{1-p_i} \\ \frac{\partial F_\alpha}{\partial p_i} &= -\frac{1-\alpha}{1-p_i} F_\alpha + \frac{\pi}{(1-p_i)^2} x_i^{-\alpha}\end{aligned}\quad (38)$$

Substitution of the above into (37) yields

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{g_i}{1-p_i} - \lambda_i + \nu_i, \quad (39)$$

where $\mathbf{g} = (g_i, i \in [n])$ has components

$$g_i \equiv -(1-\alpha)F_\alpha + \pi_i x_i^{-\alpha} + \mu_\theta(T - \pi_i). \quad (40)$$

The quantity g_i has the following important property: if $k \in [n]$ is such that $0 < p_k < 1$ then stationarity and complementary slackness require $\frac{\partial \mathcal{L}}{\partial p_k} = \lambda_k = \nu_k = 0$, which in turn requires $g_k = 0$. Next fix two distinct indices, i_1 and i_2 , such that $0 < p_{i_1}, p_{i_2} < 1$, which by the above argument, requires $g_{i_1} = g_{i_2} = 0$. Substituting (40) into this equation, substituting the earlier expressions for π_i and x_i , and solving for μ_θ yields:

$$\mu_\theta(i_1, i_2) \equiv \frac{(1-p_{i_1})f_1(p_{i_2}; \alpha) - (1-p_{i_2})f_1(p_{i_1}; \alpha)}{\pi^\alpha(p_{i_2} - p_{i_1})}, \quad (41)$$

where

$$f_1(y; \alpha) \equiv \left(\frac{1}{y} - 1\right)^\alpha \text{ for } y \in (0, 1). \quad (42)$$

Here $\mu_\theta(i_1, i_2)$ denotes the unique value of the Lagrange multiplier μ_θ enforced by the KKT conditions for indices i_1, i_2 .

As, by assumption, $|\mathcal{V}(\mathbf{p})| > 2$, there exist at least three distinct indices $\{j, k, l\}$ with $0 < p_j < p_k < p_l < 1$. As there can only be one value for μ_θ , it follows that $\mu_\theta(j, k) = \mu_\theta(j, l) = \mu_\theta(k, l)$. Equating $\mu_\theta(j, k) = \mu_\theta(j, l)$ and simplifying gives

$$(p_k - p_j)f_1(p_l; \alpha) + (p_l - p_k)f_1(p_j; \alpha) = (p_l - p_j)f_1(p_k; \alpha). \quad (43)$$

The assumed ordering of p_j, p_k, p_l ensures that p_k may be written as a convex combination of (p_j, p_l) , i.e., $p_k = t \cdot p_l + (1-t) \cdot p_j$ for

$$t = t(p_j, p_k, p_l) \equiv \frac{p_k - p_j}{p_l - p_j}, \quad 1-t = \frac{p_l - p_k}{p_l - p_j}. \quad (44)$$

By the assumptions on p_j, p_k , and p_l , both t and $1-t$ are in $(0, 1)$. Substitution of the above into (43) yields:

$$t \cdot f_1(p_l; \alpha) + (1-t) \cdot f_1(p_j; \alpha) = f_1(t \cdot p_l + (1-t) \cdot p_j; \alpha). \quad (45)$$

To summarize thus far, the KKT conditions applied to these three distinct nonzero values require each of the three pairs of indices to agree on the value of the Lagrange multiplier μ_θ (41), and this is equivalent to the condition that (45) holds for t in (44). The natural interpretation of (45) is that the function $f_1(y; \alpha)$ has the property that the convex

combination, with parameter t , of the values $f_1(p_l)$ and $f_1(p_j)$ equals the value of f_1 at the convex combination of the arguments p_j and p_l with the same parameter t . Geometrically, this requires the point $(p_k, f_1(p_k))$ to lie on the chord connecting $(p_j, f_1(p_j))$ with $(p_l, f_1(p_l))$, as illustrated in Fig. 9 (left).

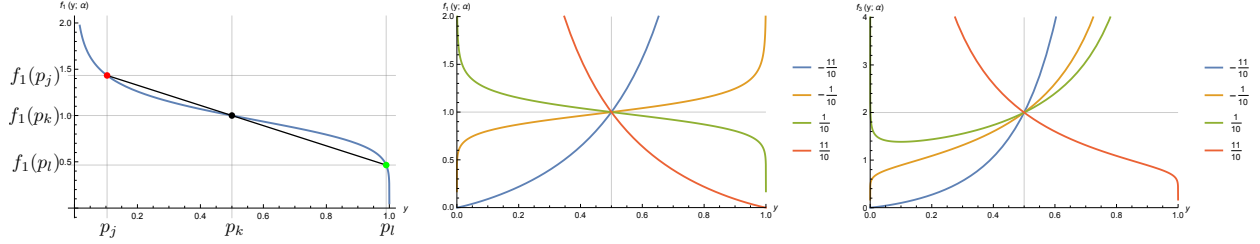


Fig. 9. Proof of Prop. 4. **Left:** an optimal \mathbf{p} with three distinct nonzero values $0 < p_j < p_k < p_l < 1$ must satisfy (45), which states the point $(p_k, f_1(p_k; \alpha))$ must lie on the chord connecting $(p_j, f_1(p_j; \alpha))$ with $(p_l, f_1(p_l; \alpha))$, for $f_1(y; \alpha)$ in (42). Here $\alpha = 1/6$ and the coordinates of the three points are (approximately) $(0.1036, 1.4329)$ (red), $(1/2, 1)$ (black), and $(0.99, 0.4649)$ (green). **Middle:** the function $f_1(y; \alpha)$ for various α ; we established f_1 to be strictly convex for $\alpha \in (-\infty, -1] \cup [1, \infty)$, precluding \mathbf{p} to be optimal for all such α . **Right:** the function $f_3(y; \alpha)$ for various α ; we established f_3 to be strictly monotone for $\alpha \in (-\infty, -1] \cup [1, \infty)$, precluding the throughput inequality to be loose for all such α .

Recall a univariate function f is strictly convex if its domain $\text{dom} f$ is convex and

$$f(sy_1 + (1-s)y_2) < sf(y_1) + (1-s)f(y_2), \quad \forall y_1, y_2 \in \text{dom} f, \quad \forall s \in (0, 1), \quad (46)$$

and is strictly concave if the inequality is reversed. In particular, the above strict inequality, for both strictly convex and strictly concave functions, ensures (45) cannot hold for any t , and thus a contradiction is reached in the assumed optimality of the \mathbf{p} with three or more distinct values, for any α for which $f_1(y; \alpha)$ is strictly convex or strictly concave. Our analysis is inconclusive in the regime where $f_1(y; \alpha)$ is neither strictly convex nor strictly concave: it may or may not be possible to satisfy (45).

This motivates us to investigate the convexity / concavity of the function $f_1(y; \alpha)$ in y . The second derivative (w.r.t. y) is

$$f_1^{(2)}(y; \alpha) = y^{-4} (y^{-1} - 1)^{\alpha-2} f_2(\alpha; y) \quad (47)$$

for

$$f_2(\alpha; y) \equiv \alpha(1 + \alpha - 2y). \quad (48)$$

Since the domain of y is $(0, 1)$, the sign of $f_1^{(2)}(y; \alpha)$ is determined by $f_2(\alpha; y)$, which we view as a quadratic in α with parameter y . Recall $f_1^{(2)}(y; \alpha) \geq 0$ is a sufficient condition for $f_1(y; \alpha)$ to be strictly convex (concave) in y . Define the sets

$$\begin{aligned} \mathcal{A}_{f_2} &= \{\alpha : f_2(\alpha; y) > 0 \quad \forall y \in (0, 1)\} \\ \mathcal{A}_{f_2}^+ &= \{\alpha > 0 : f_2(\alpha; y) > 0 \quad \forall y \in (0, 1)\} \\ \mathcal{A}_{f_2}^- &= \{\alpha < 0 : f_2(\alpha; y) > 0 \quad \forall y \in (0, 1)\} \end{aligned} \quad (49)$$

and note $f_1(y; \alpha)$ is strictly convex in y for $\alpha \in \mathcal{A}_{f_2}$. Next, observe $\mathcal{A}_{f_2} = \mathcal{A}_{f_2}^+ \cup \mathcal{A}_{f_2}^-$, since $f_2(0; y) = 0$. Furthermore, it is evident that $\mathcal{A}_{f_2}^+ = [1, \infty)$ and $\mathcal{A}_{f_2}^- = (-\infty, -1]$, and so $\mathcal{A}_{f_2} = (-\infty, -1] \cup [1, \infty)$.

Similarly it can be verified there is no value of $\alpha \in \mathbb{R}$ for which $f_2(\alpha; y) < 0$ for all $y \in (0, 1)$, meaning $f_1(y; \alpha)$ is not strictly concave on $(0, 1)$ for any α . In summary, we've established the impossibility of an optimal \mathbf{p} having $|\mathcal{V}(\mathbf{p})| > 2$ for $\alpha \in (-\infty, -1] \cup [1, \infty)$, as illustrated in Fig. 9 (middle).

We next consider the case of a throughput equality constraint, $T(\mathbf{x}(\mathbf{p})) = \theta$. The only change in the KKT conditions from the inequality constraint case is that now the sign of the Lagrange multiplier μ_θ is unrestricted. However, observe that the above proof for the inequality constraint case does not rely upon the dual feasibility condition of μ_θ . As such, the above proof holds in this case as well.

Proof of ii). By assumption that the optimizer \mathbf{p}^* has $|\mathcal{V}(\mathbf{p}^*)| = 2$, we denote the two nonzero component values by $0 < p_k < p_l < 1$. We prove by contradiction. Assuming the throughput constraint does not hold with equality namely $T(\mathbf{x}(\mathbf{p}^*)) > \theta$, it follows that the corresponding Lagrange multiplier μ_θ is zero, and in particular we must have $\mu_\theta(k, l) = 0$ in (41). This expression may be rearranged as $f_3(p_k; \alpha) = f_3(p_l; \alpha)$, for

$$f_3(y; \alpha) \equiv \frac{f_1(y; \alpha)}{1 - y}. \quad (50)$$

We next establish that $f_3(y; \alpha)$ is strictly monotone in $y \in (0, 1)$ for all $\alpha \in (-\infty, -1] \cup [1, \infty)$, as illustrated in Fig. 9. This strict monotonicity means it is impossible to have $0 < p_k < p_l < 1$ and $f_3(p_k; \alpha) = f_3(p_l; \alpha)$. The first derivative of f_3 (w.r.t. y) is

$$f_3^{(1)}(y; \alpha) = \frac{y - \alpha}{y(1 - y)^2} f_1(y; \alpha). \quad (51)$$

And thus f_3 is either always strictly monotone increasing (when $\alpha \in (-\infty, -1]$) or always strictly monotone decreasing in y (when $\alpha \in [1, \infty)$), for all $y \in (0, 1)$. This implies $\mu_\theta(k, l) = 0$ cannot hold, which in turn implies, as a consequence of complementary slackness, at an optimizer \mathbf{p}^* that has the property that $|\mathcal{V}(\mathbf{p}^*)| = 2$, the throughput inequality constraint must be tight i.e., $T(\mathbf{x}(\mathbf{p}^*)) = \theta$.

Note that in all the above analysis, the expression for the $\alpha \neq 1$ case of F_α , defined in (8), is used. As the claimed regime of α (i.e., $(-\infty, -1] \cup [1, \infty)$) to which the assertion of this proposition applies includes $\alpha = 1$, it is necessary to verify it also holds for this case. This is done separately below. ■

Proof of Prop. 4 for the $\alpha = 1$ case: We prove the two parts in the order they are given.

Proof of i). The domain $\mathbf{p} \in [0, 1]^n$ allows us to rule out the possibility of any component $p_i = 1$. We will further dismiss the case when there exists some component $p_i = 0$, because if any such zero component exists in \mathbf{p} , then the corresponding rate $x_i = 0$, which gives the objective $F_1(\mathbf{x}(\mathbf{p})) = -\infty$ meaning it is uninteresting/infeasible if we were to minimize/maximize $F_1(\mathbf{x})$. Let \mathbf{p} obey $|\mathcal{V}(\mathbf{p})| > 2$; we will show any such point cannot satisfy the KKT conditions.

We first consider the case of a throughput inequality constraint, $T(\mathbf{x}(\mathbf{p})) \geq \theta$. Since $F_1(\mathbf{x})$ is maximized iff $\tilde{F}_1(\mathbf{x}) \equiv \prod_{j=1}^n x_j$ is maximized (for $\mathbf{x} > \mathbf{0}$), we work with \tilde{F}_1 . Introduce Lagrange multipliers μ_θ , $\boldsymbol{\lambda}$, and $\boldsymbol{\nu}$, and form exactly the same Lagrangian (35), with the objective replaced by \tilde{F}_1 .

As $0 < p_i < 1$ it follows that $\lambda_i = \nu_i = 0$. As $0 < p_i < 1$ holds for all $i \in [n]$, it follows that $\pi(\mathbf{p}) \neq 0$ (defined in (36)), and as such the stationarity equation $\frac{\partial \mathcal{L}}{\partial p_i} = 0$ of (35) may be solved for μ_θ :

$$\mu_\theta = -\frac{n - \frac{1}{p_i}}{\frac{1}{1-p_i} - A(\mathbf{p})} \prod_{j=1}^n p_j (1-p_j)^{n-2}, \quad (52)$$

where $A(\mathbf{p}) \equiv \sum_{j=1}^n \frac{p_j}{1-p_j}$.

Fixing indices i_1, i_2 with $0 < p_{i_1} < p_{i_2} < 1$, the two equations $\frac{\partial \mathcal{L}}{\partial p_{i_1}} = 0$ and $\frac{\partial \mathcal{L}}{\partial p_{i_2}} = 0$ may each be solved for μ_θ in (52), equated with each other, and the resulting equation may be solved for $A(\mathbf{p})$:

$$A(\mathbf{p}) = A(i_1, i_2) = \frac{1 - np_{i_1}p_{i_2}}{(1-p_{i_1})(1-p_{i_2})}. \quad (53)$$

Here $A(i_1, i_2)$ denotes the value of $A(\mathbf{p})$ obtained from the KKT stationarity condition for indices i_1, i_2 .

Now consider three distinct indices $\{j, k, l\}$ with $0 < p_j < p_k < p_l < 1$. As there can only be one value for A , it follows that $A(j, k) = A(j, l) = A(k, l)$. Equating any pair out of these three and simplifying yields $p_s = 1/n$ where s is the common index in the two pairs of indices. Collectively this implies $p_j = p_k = p_l = 1/n$, which is a contradiction. This shows $|\mathcal{V}(\mathbf{p})| \leq 2$.

We now consider the case of a throughput equality constraint, $T(\mathbf{x}(\mathbf{p})) = \theta$. Since in this case there is no restriction on the sign of the corresponding Lagrange multiplier μ_θ , the above proof holds as well.

Proof of ii). For the second part of the proposition, we prove by contradiction. Given $|\mathcal{V}(\mathbf{p}^*)| = 2$, meaning \mathbf{p}^* has components p_k, p_l satisfying $0 < p_k < p_l < 1$, if the throughput inequality constraint is not tight at \mathbf{p}^* , then due to complementary slackness it follows $\mu_\theta = 0$, which would imply $p_k = p_l = 1/n$, a contradiction. ■

APPENDIX II

PROOFS FROM §IV

Proofs from §IV-A, §IV-B, and §IV-C are given in Appendix II-A, Appendix II-B, and Appendix II-C, respectively.

A. Proofs from §IV-A

Proof of Prop. 6: We establish the two statements in the order they are given.

Proof of 1). Recall the implicit definition of $p_s(k, n', \theta)$ in (17) in Prop. 3 enables us to write $T(p_s(k, n', \theta), k, n') = \theta$. Note first that θ is held constant in Prop. 6. Moreover, in the proof of 1) we furthermore hold n' constant, while in the proof of 2) we instead hold $n_l = n' - k$ constant. Because of this, we suppress in the proof of 1) the dependence on both θ and n' , and in particular, $p_s(k)$ is defined as the unique solution, when it exists, to the equation $T(p_s(k), k) = \theta$, and $F_{-1}(p_s(k, n', \theta), k, n')$ (defined in (22)) is written as $F_{-1}(p_s(k), k)$. It is convenient to treat k as a continuous variable in what follows, i.e., to replace $k \in \{1, \dots, n' - 1\}$ with $k \in [1, n' - 1]$. Note here we write $p_s(k)$ because the throughput equality constraint (implicitly) determines p_s as a function of k under the (p_s, k, n') parameterization. It is straightforward to establish $\frac{\partial}{\partial p_s} T(p_s, k) \neq 0$ over the domain of (p_s, k) , and as such we can apply the implicit function theorem:

$$\frac{d}{dk} p_s(k) = -\frac{\frac{\partial}{\partial k} T(p_s, k)}{\frac{\partial}{\partial p_s} T(p_s, k)}. \quad (54)$$

The *total derivative*⁷ of F_{-1} w.r.t. k is

$$\frac{d}{dk}F_{-1}(p_s(k), k) = \frac{\partial}{\partial k}F_{-1}(p_s, k) + \frac{\partial}{\partial p_s}F_{-1}(p_s, k) \frac{d}{dk}p_s(k). \quad (55)$$

Computing and substituting the three derviatives in the above expression yields:

$$\frac{d}{dk}F_{-1}(p_s(k), k) = \frac{(1-p_s)^{2(k-1)}(1-p_l)^{2(n'-k)}}{2(kp_s + n' - k - 1)^2} f_1(p_s, k), \quad (56)$$

where

$$\begin{aligned} f_1(p_s, k) \equiv & (n'p_s - 1)(-2k(p_s - 1) + n'(p_s - 2) + 1) + \\ & 2(p_s - 1)(k - n')(kp_s + n' - k - 1) \log \frac{1 - p_l}{1 - p_s}. \end{aligned} \quad (57)$$

It is evident from (56) that showing $F_{-1}(p_s(k), k)$ to be increasing in k is equivalent to showing $f_1(p_s, k) > 0$. After rearrangement, it may be seen that showing $f_1(p_s, k) > 0$ is equivalent to showing

$$f_3(p_s, k) < f_2(p_s, k), \quad (58)$$

where

$$f_3(p_s, k) \equiv \log \left(1 + \frac{p_l - p_s}{1 - p_l} \right), \quad (59)$$

and

$$f_2(p_s, k) \equiv \frac{(n'p_s - 1)(-2k(p_s - 1) + n'(p_s - 2) + 1)}{2(p_s - 1)(k - n')(kp_s + n' - k - 1)}. \quad (60)$$

In F_{-1}, f_1, f_2, f_3 above the variable p_s is not in fact free, but instead is determined by $T(p_s(k), k) = \theta$. Below, we show a stronger result that in fact (58) holds for *all* $k \geq 0$ and for *all* $p_s \in (0, 1/n')$. Our approach to showing (58) is as follows: to show two univariate functions $g_1(x), g_2(x)$ with domain \mathbb{R}_+ are ordered as $g_1(x) < g_2(x)$ for all x , it suffices to show *i)* $g'_1(x) \leq g'_2(x)$ and *ii)* $g_1(0) < g_2(0)$ (which can be easily verified by working with a new function $g_2(x) - g_1(x)$). The first step towards (58) is to establish the ordering of the derivatives. Recalling $p_l = p_l(p_s, k, n')$ (12), define

$$z = z(p_s, k) \equiv \frac{p_l - p_s}{1 - p_l} = \frac{1 - n'p_s}{n' - 1 - k + kp_s} > 0, \quad (61)$$

substitute z into (59), and observe:

$$\Delta(p_s, k) = \frac{(n'p_s - 1)^3}{2(p_s - 1)(k - n')^2(kp_s + n' - k - 1)^2} > 0, \quad (62)$$

for $\Delta(p_s, k) \equiv \frac{\partial}{\partial k}f_2(p_s, k) - \frac{\partial}{\partial k}f_3(p_s, k)$. The second step towards (58) is to establish $f_3(p_s, 0) < f_2(p_s, 0)$. In fact we show

$$f_3(p_s, 0) < f_4(z(p_s, 0)) < f_2(p_s, 0), \quad (63)$$

⁷In this case, some authors such as Chiang and Wainwright [28] may call this *partial total derivative* and use a different notation (see discussion toward the end of Section 8.4). It is “partial” because the function (F_{-1}) by definition still depends on another exogenous variable (n'); it is “total” in that it fully captures both the direct and indirect influence of k .

for $f_4(z) \equiv z - \frac{1}{2}z^2 + \frac{1}{3}z^3$. The first inequality in (63) follows from the series expansion of $\log(1+z)$ and valid for all $z > 0$. The second inequality in (63) is established by computing

$$f_2(p_s, 0) - f_4(z(p_s, 0)) = \frac{(n'p_s - 1)^3(2n'p_s + n' - 3)}{6(n' - 1)^3n'(p_s - 1)}, \quad (64)$$

which is positive for all $n' \geq 3$. Note $n' \geq 2$ since $\mathbf{p} \in \partial\mathcal{S}_2$, and the $n' = 2$ case can be skipped as $k = 1$ always holds. This concludes the proof of the first part of the proposition.

Proof of 2). In the second statement of Prop. 6 we again hold θ constant, but instead of also holding n' constant (as in the first statement), we now hold n_l constant, where n_l is the number of components in $\mathbf{p} \in \partial\mathcal{S}_2$ taking (the larger) value p_l . It is clear that we can just as easily parameterize $\mathbf{p} \in \partial\mathcal{S}_2$ using the three free parameters $[p_s, k, n_l]$ as with (p_s, k, n') (the change in parameterization emphasized by the change from parentheses to square braces) using the mapping $k + n_l = n'$ (with p_s and k still defined as before). The new parameters must take values such that $p_s \in (0, 1/(k + n_l))$, and $(k, n_l) \in \mathcal{D}_n$, where

$$\mathcal{D}_n \equiv \{(k, n_l) \in \mathbb{N}^2 : k \geq 1, n_l \geq 1, k + n_l \leq n\}. \quad (65)$$

We now define the functions $T[p_s, k, n_l] = T(\mathbf{x}(\mathbf{p}[p_s, k, n_l]))$ and $F_{-1}[p_s, k, n_l] = F_{-1}(\mathbf{x}(\mathbf{p}[p_s, k, n_l]))$ under this new parameterization. The throughput constraint $T[p_s, k, n_l] = \theta$ again implicitly defines a function $p_s[k, n_l, \theta]$ satisfying $T[p_s[k, n_l, \theta], k, n_l] = \theta$. Analogous to part 1) of the proof, we suppress the dependence upon n_l and θ , and again because the throughput equality constraint determines p_s as a function of k , we write $p_s[k, n_l, \theta]$ as $p_s[k]$, the throughput constraint function as $T[p_s[k], k] = \theta$, and the objective $F_{-1}[p_s[k, n_l, \theta], k, n_l]$ as $F_{-1}[p_s[k], k]$.

It is straightforward to establish $\frac{\partial}{\partial p_s} T[p_s, k] \neq 0$ over the domain of (p_s, k) , and as such we can apply the implicit function theorem (which again treats k as a continuous variable):

$$\frac{d}{dk} p_s[k] = - \frac{\frac{\partial}{\partial k} T[p_s, k]}{\frac{\partial}{\partial p_s} T[p_s, k]}. \quad (66)$$

The *total derivative* of F_{-1} w.r.t. k is

$$\frac{d}{dk} F_{-1}[p_s[k], k] = \frac{\partial}{\partial k} F_{-1}[p_s, k] + \frac{\partial}{\partial p_s} F_{-1}[p_s, k] \frac{d}{dk} p_s[k]. \quad (67)$$

Computing and substituting the above derivatives yields

$$\frac{d}{dk} F_{-1}[p_s[k], k] = \frac{1}{2n_l} (1 - p_s)^{2(k-1)} \left(\frac{kp_s + n_l - 1}{n_l} \right)^{2n_l-1} f_5[p_s, k], \quad (68)$$

where

$$f_5[p_s, k] \equiv -kp_s^3 + (n_l + 1)p_s^2 - 2n_l p_s - 2n_l(1 - p_s) \log(1 - p_s), \quad (69)$$

and the sign of the derivative is easily seen to equal the sign of the above function. Thus part 2) of the proposition is established by showing $f_5[p_s, k] > 0$ for $k \in [n - n_l]$ and $p_s \in (0, 1/(k + n_l))$. Using the upper bound $\log(1 - p_s) \leq (-p_s) - \frac{1}{2}(-p_s)^2$ we obtain

$$f_5[p_s, k] \geq p_s^2(1 - p_s(k + n_l)) > 0. \quad (70)$$

This concludes the proof of the second part of the proposition. ■

B. Proofs from §IV-B

Proof of Thm. 1: There are three regimes for θ given in Thm. 1. The proof consists of two parts: part *i*) addresses regime 1, while part *ii*) addresses regimes 2) and 3).

Part i) (Regime 1)). The claim here is that the maximum fairness of 1 is achievable, attained when all the x_i 's are equal to θ/n . It is not hard to see all the x_i 's are equal iff all the associated controls p_i 's (i.e., satisfying (2)) are equal, in which case $\theta/n = x_i = p(1-p)^{n-1}$ for each $i \in [n]$, for some $p \in [0, 1]$ to be determined. The existence of such a p follows from Lem. 1 and thus the claim is proved.

Part ii) (Regimes 2) and 3)). This part of the proof is divided into three steps. Recall \mathbf{p}^* denotes the optimal control.

Step 1: $\mathbf{p}^* \in \partial\mathcal{S}$. That \mathbf{p}^* must be a probability vector follows from Cor. 1 in §III-A.

Step 2: $\mathbf{p}^* \in \partial\mathcal{S}_{1,2}$. By Prop. 4 in §III-B, $|\mathcal{V}(\mathbf{p}^*)| \leq 2$, as the minimization problem (20) is a special case of the extremization problem (19) in Prop. 4 with $\alpha = -1$. Then together with $\mathbf{p}^* \in \partial\mathcal{S}$, it gives $\mathbf{p}^* \in \partial\mathcal{S}_{1,2}$.

Step 3: Following Remark 2, regimes 2) and 3) are grouped together meaning the target throughput $\theta \in [\theta_t, \theta_{t-1})$. By item 3) in Prop. 3, the set of feasible (k, n') pairs for which there exists a $\mathbf{p} \in \partial\mathcal{S}_{1,2}$ satisfying $T(\mathbf{x}(\mathbf{p})) = \theta$ is the set $\mathcal{D}_{t,n}$ in (18), illustrated in Fig. 1.

Case 1 : assuming $\mathbf{p}^* \in \partial\mathcal{S}_2$, we can then apply the two monotonicity properties stated in Prop. 6 to the set $\mathcal{D}_{t,n}$, which shows the optimal $(k^*, n'^*) = (1, t)$. Applying $(k, n') = (1, t)$ to the throughput constraint equation (17) yields (24). Furthermore, as $p_s^* \in (0, 1/n'^*)$ this in turn shows (due to the monotonicity established in item 1) of Prop. 3) the achievable throughput range by varying p_s is the open interval (θ_t, θ_{t-1}) .

Case 2 : assuming $\mathbf{p}^* \in \partial\mathcal{S}_1$, we let such a \mathbf{p}^* be parameterized by n' (Def. 1). The corresponding extremizer in the rate space is $\mathbf{x}^* = \mathbf{x}^*(n') \equiv \frac{\theta_{n'}}{n'} \sum_{i=1}^{n'} \mathbf{e}_i$. Satisfying the feasibility constraint for $\theta \in [\theta_t, \theta_{t-1})$ requires $n' \leq t$, and in fact n' can only equal t due to its integer support. This shows the optimal $n'^* = t$ (thus $\mathbf{p}^* = (1/t) \sum_{i=1}^t \mathbf{e}_i$ and $F_j^* = t/n$). Furthermore, this in turn shows if $\theta = \theta_t$ then the corresponding $\mathbf{p}^* \in \partial\mathcal{S}_1$.

Clearly the target throughput range $[\theta_t, \theta_{t-1})$ is partitioned as $(\theta_t, \theta_{t-1}) \cup \{\theta_t\}$ where the extremizers for the former (regime 3)) and latter (regime 2)) are found in cases 1 and 2 respectively.

Finally, $\tilde{T}(F)$ in (23) is obtained by observing the above results for regime 2 as n points $\{(T_t, F_t)\}_{t \in [n]}$ on the throughput–fairness tradeoff plot, with $T_t = \theta_t$ and $F_t = t/n$. Thus, to interpolate the n points via a function $\tilde{T}(F)$ it suffices to use $\tilde{T}(F) = T_{nF}$ and treat F as a continuous variable. ■

Proof of Thm. 2: Part *i*) (Regime 1)). In the proof of Thm. 1 it is shown that for this regime, the maximum fairness 1 can be attained with the throughput constraint satisfied with equality. This continues to hold here.

Part *ii*) (Regimes 2) and 3)). The second and third regimes namely the case when $\theta \geq \theta_n$.

Step 1: $\mathbf{p}^* \in \partial\mathcal{S}$. This is because the global minimizer must lie on a hyperplane $\mathcal{H}_{\theta^*} = \{\mathbf{x} : \sum_i x_i = \theta^*\}$ for some $\theta^* \geq \theta$. Then the same step in the proof of Thm. 1 applies.

Step 2: $\mathbf{p}^* \in \partial\mathcal{S}_{1,2}$. The same step in the proof of Thm. 1 applies, as the extremization problem (19) in Prop. 4 includes the case of throughput inequality constraint.

Step 3 is divided into two sub-steps, one for each regime. Recall $\partial\mathcal{S}_{1,2}$ is the disjoint union of $\partial\mathcal{S}_1$ and $\partial\mathcal{S}_2$, and n' denotes the number of nonzero component(s) of \mathbf{p}^* .

Regime 2): when $\theta = \theta_t$ for some $t \in [n]$.

Case 1: *assuming* $\mathbf{p}^* \in \partial\mathcal{S}_2$, since Item *ii*) of Prop. 4 says under the assumption $|\mathcal{V}(\mathbf{p}^*)| = 2$, an extremizer has to satisfy the throughput constraint with equality, this justifies we can apply Thm. 1 (regime 2)). Doing so gives the extremizer as $\mathbf{p}^* = (1/t) \sum_{i=1}^t \mathbf{e}_i$. But this contradicts our assumption that $\mathbf{p}^* \in \partial\mathcal{S}_2$.

Case 2: *assuming* $\mathbf{p}^* \in \partial\mathcal{S}_1$, it follows that $\mathbf{x}(\mathbf{p}^*) = \frac{\theta_{n'}}{n'} \sum_{i=1}^{n'} \mathbf{e}_i$. On one hand, $T(\mathbf{x}(\mathbf{p}^*)) \geq \theta$ for such an \mathbf{x} requires $n' \leq t$; on the other hand, the objective $F_{-1}(\mathbf{x}(\mathbf{p}^*))$, to be minimized, is decreasing in n' . Together they imply the optimal $n'^* = t$, with the corresponding fairness $F_J^* = t/n$.

Therefore, the extremizer for $\theta = \theta_t$ actually comes from $\partial\mathcal{S}_1$ and is given by $\mathbf{p}^* = (1/t) \sum_{i=1}^t \mathbf{e}_i$ with $F_J^* = t/n$.

Regime 3): when $\theta \in (\theta_t, \theta_{t-1})$ for some $t \in \{2, \dots, n\}$.

Case 1: *assuming* $\mathbf{p}^* \in \partial\mathcal{S}_1$: similar to what we have done above, satisfying the feasibility constraint $T(\mathbf{x}) \geq \theta$ requires $n' \leq t - 1$, while the objective function $F_{-1}(\mathbf{x})$ is decreasing in n' which means n' is desired to be as large as possible. Together they imply the optimal $n'^* = t - 1$, with the corresponding fairness $F_J^* = (t - 1)/n$.

Case 2: *assuming* $\mathbf{p}^* \in \partial\mathcal{S}_2$: again item *ii*) of Prop. 4 justifies Thm. 1 (regime 3)) is applicable. Furthermore, in this case, the optimal solution \mathbf{p}^* from $\partial\mathcal{S}_2$ is such that $F_J^* \in ((t - 1)/n, t/n)$ due to the monotonicity and continuity of the T-F tradeoff curve (Thm. 3, items 2, 4) and the just proved result for regime 2).

As the optimal solution from $\partial\mathcal{S}_2$ outperforms that from $\partial\mathcal{S}_1$, this shows the desired extremizer is indeed from $\partial\mathcal{S}_2$ and is as stated for regime 3) in Thm. 1.

In summary, the solution to the Jain throughput–fairness tradeoff (20) remains unchanged. ■

C. Proofs from §IV-C

The following lemma is essential to the proof of item 5) of Thm. 3.

Lemma 3: Given an integer $n \geq 3$, the following two polynomials in n are both positive for $p_s \in (0, 1/n)$.

$$\begin{aligned}
 f_{\text{de}}(n; p_s) &= n^2 p_s^2 + n(p_s^4 - 2p_s^3 + 2p_s^2 - 4p_s + 1) - 2p_s^2 + 4p_s - 1 \\
 f_{\text{nu}}(n; p_s) &= n^4(p_s^5 - p_s^4 - 2p_s^3 + 5p_s^2 - 4p_s) + \\
 &\quad n^3(p_s^7 + p_s^6 - 12p_s^5 + 19p_s^4 - 10p_s^3 - 6p_s^2 + 6p_s + 5) + \\
 &\quad n^2(2p_s^7 - 16p_s^6 + 41p_s^5 - 55p_s^4 + 45p_s^3 - 20p_s^2 + 17p_s - 20) + \\
 &\quad n(7p_s^4 - 18p_s^3 + 24p_s^2 - 35p_s + 26) \\
 &\quad - 6p_s^2 + 16p_s - 11.
 \end{aligned} \tag{71}$$

Proof of Lem. 3: In both parts of the proof we treat n as a continuous variable and view p_s as fixed.

Part i) ($f_{\text{de}}(n; p_s) > 0$). We prove this by showing $f_{\text{de}}(3; p_s) > 0$ and $\frac{d}{dn} f_{\text{de}}(n; p_s) > 0$ for all $n \geq 3$. First, $f_{\text{de}}(3; p_s) = 3p_s^4 - 6p_s^3 + 13p_s^2 - 8p_s + 2$. Since this quartic (in p_s) has all its four roots being complex, this means this polynomial (in p_s) is either always positive or always negative for all $p_s \in \mathbb{R}$. We can test this by setting

$p_s = 0$ and this shows its positiveness. Second, $\frac{d}{dn} f_{de}(n; p_s) = 2np_s^2 + p_s^4 - 2p_s^3 + 2p_s^2 - 4p_s + 1$, which is lower bounded by $p_s^4 - 2p_s^3 + 8p_s^2 - 4p_s + 1$ since $n \geq 3$. Again this quartic (in p_s) can be shown to have all its four roots being complex and we can use any specific value of $p_s \in \mathbb{R}$ to verify its positiveness.

Part ii) ($f_{nu}(n; p_s) > 0$). The condition $p_s \in (0, 1/n)$ for $n \geq 3$ then translates to $n \in [3, 1/p_s)$. We will focus on showing $f_{nu}(n; p_s)$ as a polynomial in n does not have any real root on $n \in [3, 1/p_s)$, which suggests $f_{nu}(n; p_s)$ is either always positive or always negative on this interval and we then only need to test this out using any specific point in the interval. A plot of $f_{nu}(n; p_s)$ versus n for fixed p_s is shown in Fig. 10. In the following we will show a slightly stronger result, namely to extend the domain of interest to $(2, 1/p_s)$. For notational simplicity we let $p_s = 1/m$ for $m > n$ and express the coefficients of the polynomial $f_{nu}(n; p_s)$ using m , and we will also use the shorter notation $f_{nu}(n)$. The j^{th} derivative (w.r.t. n) of f_{nu} is denoted $f_{nu}^{(j)}(n) \equiv \frac{d^j}{dn^j} f_{nu}(n)$.

We use the Budan-Fourier theorem, which (partially) characterizes the number of real roots of a polynomial in any given interval. Specifically, let $v(a)$ and $v(b)$ denote the number of sign changes (i.e., sign variation) of the Fourier sequence $\{f_{nu}(n), f_{nu}^{(1)}(n), \dots, f_{nu}^{(4)}(n)\}$ when $n = a$ and b respectively, for $a < b$. This theorem says the number of real roots in (a, b) , each root counted with proper multiplicity, equals $v(a) - v(b)$ minus an even nonnegative integer.

We can verify $v(2) = 1$ since the signs of the Fourier sequence are $+$ $+$ $+$ \mp $-$ (note the sign of $f_{nu}^{(3)}(2)$ is undetermined, if we only know $m > 3$). We can further verify $v(m) = 1$ since the signs of the Fourier sequence are $+$ $-$ $-$ $-$ $-$. Since $v(2) - v(m)$ already equals 0, applying Budan-Fourier theorem, we see the polynomial $f_{nu}(n)$ has no real root on $(2, m)$.

The Fourier sequence at $a = 2$ and $b = m$ are given below in a form that facilitates checking their sign. Namely:

$$\begin{aligned} m^7 f_{nu}(2) &= (m-2)(m^4(m^2-6) + m^2(20m-30) + (24m-8)) \\ m^7 f_{nu}^{(1)}(2) &= (m^3(6m^4 - 23m^3 + 32m^2 - 22m - 17) + m(52m - 52) + 20) \\ m^7 f_{nu}^{(2)}(2) &= 2(m^4(10m^3 - 43m^2 + 64m - 63) + m(35m^2 - 7m - 10) + 8) \\ m^7 f_{nu}^{(3)}(2) &= 6(5m^7 - 26m^6 + 34m^5 - 26m^4 + 11m^3 - 4m^2 + m + 1) \\ m^5 f_{nu}^{(4)}(2) &= -24(m^3(4m-5) + (2m^2 + m - 1)) \end{aligned} \quad (72)$$

and

$$\begin{aligned} m^5 f_{nu}(m) &= (m-2)(m-1)^7 \\ m^6 f_{nu}^{(1)}(m) &= -(m-1)^5(m^3 + m(7m-9) + 4) \\ m^7 f_{nu}^{(2)}(m) &= -2(m-1)^3(m^3(9m^2 - m - 17) + m(17m-7) + 2) \\ m^7 f_{nu}^{(3)}(m) &= -6(m^6(11m-26) + 14m^5 + m^3(14m-23) + (12m^2 - m - 1)) \\ m^5 f_{nu}^{(4)}(m) &= -24(m^3(4m-5) + (2m^2 + m - 1)). \end{aligned} \quad (73)$$

Since $m = 1/p_s > n \geq 3$, it is not hard to verify the sign of the terms grouped by inner parentheses to be positive (and hence determine $v(2)$ and $v(m)$), except for $f_{nu}^{(3)}(2)$, but, as mentioned, this sign does not affect the value of

$v(2)$.

It remains to use any specific point on $(2, m)$ to determine the sign of $f_{\text{nu}}(n)$ over the entire $(2, m)$, e.g.,

$$f_{\text{nu}}(3) = \frac{1}{m^7} (m^3(22m^4 - 98m^3 + 129m^2 - 81m - 42) + (126m^2 - 117m + 45)). \quad (74)$$

It can be verified that the quartic and quadratic enclosed by the two pairs of inner parentheses in the above expression are both positive. This shows f_{nu} is positive at $n = 3$, and as argued above, this proves f_{nu} is positive over $n \in (2, 1/p_s)$. ■

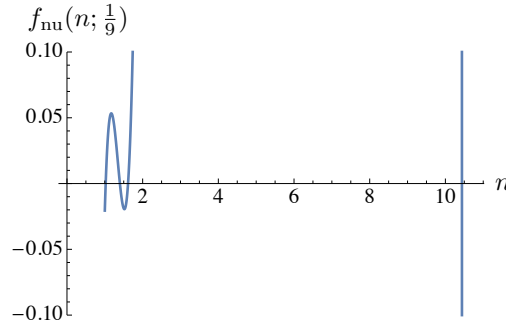


Fig. 10. $f_{\text{nu}}(n; p_s)$ when $p_s = 1/9$; the top part is not shown, in order to better view all the roots. Three of them are between 1 and 2 and the remaining one is in $(1/p_s, \infty)$. As $f_{\text{nu}}(n; p_s)$ is a 4th order polynomial in n , it has a total of four roots and hence no root exists in the interval $(2, 1/p_s)$.

Proof of Thm. 3: We write $F_J^*(\theta; n)$ to denote the optimized Jain's fairness under a throughput constraint $T(\mathbf{x}) = \theta$, where n serves as a parameter but not a free variable in the optimization.

The feasible set Λ is parameterized by \mathbf{p} via (2), and when $\theta \in [\theta_t, \theta_{t-1})$ for $t \in \{2, \dots, n\}$, we know from Thm. 1 the unique extremizer is characterized by the tuple (p_s^*, k^*, n'^*) , with $k^* = 1$ and $n'^* = t$, defined in Def. 1. It is clear from Thm. 1 that this tuple is a function of θ , and may be written as $(p_s^*(\theta), 1, t)$. Therefore the notation $F_J^*(\theta; n)$ should be understood as

$$F_J^*(\theta; n) \equiv F_J(\mathbf{x}(\mathbf{p}(p_s^*(\theta), 1, t))), \quad (75)$$

with F_J defined in (7). Observe also the identity

$$\theta \equiv T(p_s^*(\theta), 1, t), \quad (76)$$

for $T(p_s, k, n')$ defined in (15) and p_s^* the solution of (24). This is used to compute the dependence of p_s^* on θ .

Item 1). That $p_s^*(\theta)$ is piecewise decreasing in θ follows from (76) and Prop. 3 (item 1)):

$$\frac{dp_s^*(\theta)}{d\theta} = \left(\frac{d\theta(p_s^*)}{dp_s^*} \right)^{-1} = \left(\frac{d}{dp_s^*} T(p_s^*, 1, t) \right)^{-1} < 0, \quad (77)$$

That $p_l^*(\theta)$ is piecewise increasing in θ follows from p_l in Def. 1 and (77).

In fact, a stronger statement is that $p_l^*(\theta)$ is increasing in θ , albeit not everywhere differentiable. To see this, let us look at two adjacent active throughput intervals on the T-F plot: $[\theta_t, \theta_{t-1})$, $[\theta_{t-1}, \theta_{t-2})$. When θ sweeps over the

first interval (for which $n'^* = t$), p_s^* decreases from $1/t$ (when $\theta = \theta_t$) to 0 (when $\theta = \theta_{t-1}$) and correspondingly p_l^* increases from $1/t$ (when $\theta = \theta_t$) to $1/(t-1)$ (when $\theta = \theta_{t-1}$). Moving onto the second interval (for which $n'^* = t-1$), similarly, p_s^* (p_l^*) decreases (increases) from $1/(t-1)$ ($1/(t-1)$) to 0 ($1/(t-2)$). Clearly p_s^* is not monotonic over the entire $\theta \in (\theta_n, 1)$ whereas p_l^* is monotonic.

Next we show $p_l^*(\theta)$ is not differentiable at the boundary of active throughput intervals. More precisely, at the boundary of the two intervals $[\theta_t, \theta_{t-1})$ and $[\theta_{t-1}, \theta_{t-2})$, i.e., $\theta = \theta_{t-1}$, we compute the left- and right- derivative respectively and show they are not equal. That is, nondifferentiability at θ_{t-1} is established by showing

$$\left. \frac{d}{d\theta} p_l(p_s^*(\theta), 1, t) \right|_{p_s^*(\theta)=0} \neq \left. \frac{d}{d\theta} p_l(p_s^*(\theta), 1, t-1) \right|_{p_s^*(\theta)=\frac{1}{t-1}}, \quad (78)$$

where

$$\frac{dp_l^*(\theta)}{d\theta} = \frac{dp_l^*(\theta)}{dp_s^*} \frac{dp_s^*}{d\theta} = \frac{\frac{dp_l^*(\theta)}{dp_s^*}}{\frac{dp_s^*(\theta)}{dp_s^*}} = \frac{-\frac{k^*}{n'^*-k^*}}{\frac{d}{dp_s^*} T(p_s^*, 1, t)}, \quad (79)$$

with $T(p_s^*, 1, t)$ again coming from (76). Since the LHS of (78) equals $\left(\frac{t-2}{t-1}\right)^{2-t}$ whereas the RHS equals infinity, this establishes (78).

Next, we look at the dependence of $x_s^*(\theta)$, $x_l^*(\theta)$ upon θ . Let $\theta \in [\theta_t, \theta_{t-1})$. As $(k^*, n'^*) = (1, t)$, we have

$$x_s^* = p_s^*(1 - p_l^*)^{t-1}, \quad x_l^* = p_l^*(1 - p_s^*)(1 - p_l^*)^{t-2}. \quad (80)$$

That $\frac{dx_s^*(\theta)}{d\theta} < 0$ follows easily from $\frac{dp_s^*(\theta)}{d\theta} < 0$ and $\frac{dp_l^*(\theta)}{d\theta} > 0$. To show $\frac{dx_l^*(\theta)}{d\theta} > 0$, it can be seen from (80) that it suffices to show $p_l^*(1 - p_l^*)^{t-2}$ is increasing in p_l^* : we can verify the function $p(1-p)^{t-2}$ is increasing in p when $p \in (0, 1/(t-1))$, which includes the range of p_l^* when $\theta \in [\theta_t, \theta_{t-1})$ namely $[1/t, 1/(t-1))$. This proves $\frac{dx_l^*(\theta)}{d\theta} > 0$.

Finally, we want to show at the boundary of active throughput intervals, $x_l^*(\theta)$ is not differentiable. First, let \mathbf{p}^* be parameterized by (p_s^*, k^*, n'^*) . Applying the chain rule, we have

$$\frac{dx_l^*}{d\theta} = \frac{dx_l^*}{dp_s^*} \frac{dp_s^*}{d\theta} = \frac{\frac{d}{dp_s^*} p_l^*(1 - p_s^*)^{k^*} (1 - p_l^*)^{n'^*-k^*-1}}{\frac{d}{dp_s^*} T(p_s^*, k^*, n'^*)} = \frac{1 - p_s^*}{1 - n'^* p_s^*}. \quad (81)$$

Second, we need to show the derivative $\frac{dx_l^*}{d\theta}$ in (81) when θ is in $[\theta_t, \theta_{t-1})$ and approaches θ_{t-1} from below does not equal to this derivative when θ is in $[\theta_{t-1}, \theta_{t-2})$ and approaches θ_{t-1} from above. Therefore, similar to (78), we need to verify

$$\left. \frac{dx_l^*}{d\theta} \right|_{(0,1,t)} \neq \left. \frac{dx_l^*}{d\theta} \right|_{(\frac{1}{t-1}, 1, t-1)}. \quad (82)$$

Applying the computed result in (81), we see the LHS of (82) equals 1 while its RHS equals infinity: this shows the nondifferentiability of $x_l^*(\theta)$ at the critical throughputs.

Item 2). We claim that it suffices to show the monotone decreasing property when $\theta \in [\theta_n, \theta_{n-1})$ for each $n \geq 2$. To see this, we prove by mathematical induction. For the base case, namely when $n = 2$, there is only one active throughput interval $[\theta_2, \theta_1)$ and the monotonicity follows from the assumption. Now assuming the monotonicity holds for $n = n_0 \geq 2$ i.e., $\frac{d}{d\theta} F_J^*(\theta; n_0) < 0$ over $\theta \in [\theta_{n_0}, 1)$, we need to show it continues to hold when

$n = n_0 + 1$ i.e., $\frac{d}{d\theta} F_J^*(\theta; n_0 + 1) < 0$ over $\theta \in [\theta_{n_0+1}, 1)$. There are two cases: when $\theta \in [\theta_{n_0+1}, \theta_{n_0})$ the monotonicity follows from the assumption; when $\theta \in [\theta_{n_0}, 1)$, specializing (25) with $l = 1$, $n = n_0 + 1$ gives $F_J^*(\theta; n_0 + 1) = \frac{n_0}{n_0+1} F_J^*(\theta; n_0)$: the monotonicity then follows from the induction hypothesis. This proves the claim.

Now, let the number of users be n and $\theta \in [\theta_n, \theta_{n-1})$. Thm. 1 says $k^* = 1$, $n'^* = t = n$ and we can compute

$$\frac{d}{d\theta} F_J^*(\theta; n) = \frac{d}{dp_s^*(\theta)} F_J(p_s^*(\theta); n) \left(\frac{d\theta(p_s^*)}{dp_s^*} \right)^{-1} = \frac{d}{dp_s^*(\theta)} F_J(p_s^*(\theta); n) \left(\frac{d}{dp_s^*} T(p_s^*, 1, n) \right)^{-1}, \quad (83)$$

where the second equality comes from (76). Substituting the definition of F_J and T in (7) and (15), we get

$$\frac{d}{d\theta} F_J^*(\theta; n) = - \frac{2(1 - p_s^*)(n + p_s^* - 2)^3 \left(\frac{n + p_s^* - 2}{n - 1} \right)^{-n} (n(-p_s^*(1 - p_s^*) + 1) - 1)}{n [n^2 p_s^{*2} + n((p_s^* - 2)(p_s^{*2} + 2)p_s^* + 1) - 2(p_s^* - 2)p_s^* - 1]^2}, \quad (84)$$

which can be verified to be negative for all $n \geq 2$ and $p_s^* \in (0, 1)$. Finally the monotone decreasing property over $[\theta_n, 1)$ (namely not just piecewise) follows from continuity of the T-F curve, shown in item 4).

Item 3). Again we decompose the interval $[\theta_n, 1)$ into $[\theta_n, \theta_{n-1}) \cup [\theta_{n-1}, 1)$. When $\theta \in [\theta_n, \theta_{n-1})$ this property automatically holds because for all $n_s < n$ we have $F_J^*(\theta; n_s) \equiv 1$ since $\theta < \theta_{n-1} \leq \theta_{n_s}$. When $\theta \in [\theta_{n-1}, 1)$, specializing (25) with $l = 1$ gives $F_J^*(\theta; n) = \frac{n-1}{n} F_J^*(\theta; n-1) < F_J^*(\theta; n-1)$, which proves the desired monotone decreasing in n property. Graphically, this corresponds to the observation that as n increases, the T-F tradeoff curve will tend closer to the θ -axis. Furthermore, since the sequence $\{\theta_n\}$ is decreasing in n , the range of θ for which the maximum achievable fairness is less than 1 (namely $(\theta_n, 1)$) always extends toward the lower bound $1/e$, and thus the full curve for any given n will tend closer to the F_J^* -axis, too.

Item 4). We first prove continuity in three steps. *a)* The extremizers in regime 2 can be viewed as limiting cases of those in regime 3. *b)* Within regime 3, since the root (on the complex plane) of a polynomial equation is continuous in its coefficients [29, §3.9], and since the polynomial equation (24) only has a single real root (p_s^*) it must also be continuous. *c)* The function F_J^* in (75) is continuous in p_s^* . We next prove nondifferentiability occurs when $\theta = \theta_{n_s}$ for all n_s smaller than n . We claim it suffices to only verify this when $n_s = n - 1$ but for *all* $n \geq 3$. To see this, specializing (25) with $l = 1$ and taking the derivative w.r.t. θ gives

$$\frac{d}{d\theta} F_J^*(\theta; n) = \frac{n-1}{n} \frac{d}{d\theta} F_J^*(\theta; n-1), \quad \forall \theta \geq \theta_{n-1}. \quad (85)$$

This implies the non-differentiability will be “inherited” as n increases (by 1), and hence one can prove this claim using mathematical induction similar to what is done in the proof of item 2). Mathematically we compare the following two (scaled) derivatives and show they are not equal at the throughput boundary θ_{n-1} .

$$\left. \frac{d}{d\theta} F_J^*(\theta; n) \right|_{\theta \uparrow \theta_{n-1}} \neq \frac{n-1}{n} \left. \frac{d}{d\theta} F_J^*(\theta; n-1) \right|_{\theta \downarrow \theta_{n-1}} \quad (86)$$

Note when the number of users is n , θ_{n-1} is the right-end of its active interval $[\theta_n, \theta_{n-1})$ and is attained when $\lim_{\theta \uparrow \theta_{n-1}} p_s^*(\theta) = 0$, whereas when the number of users is $n - 1$, θ_{n-1} is the left-end of its active interval $[\theta_{n-1}, \theta_{n-2})$ and is attained when $\lim_{\theta \downarrow \theta_{n-1}} p_s^*(\theta) = 1/(n - 1)$. Therefore the LHS of (86) is given by (84) with p_s^* set to 0 while the derivative in the RHS of (86) is given by (84) with n reparameterized as $n - 1$ and p_s^* set

to $1/(n-1)$. We can verify their ratio is $(n-2)/(n-1)$ which does not equal 1, although it approaches 1 as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \frac{\frac{d}{d\theta} F_J^*(\theta; n) \big|_{\theta \uparrow \theta_{n-1}}}{\frac{n-1}{n} \frac{d}{d\theta} F_J^*(\theta; n-1) \big|_{\theta \downarrow \theta_{n-1}}} = 1 \quad (87)$$

Item 5). We claim again that it suffices to show convexity when $\theta \in [\theta_n, \theta_{n-1})$ but for *all* $n \geq 2$; the proof of this claim is similar to the one given in proving item 2): essentially (25) implies the T-F curve for θ in a non-active throughput interval may be obtained by linear scaling of some appropriate curve section for which θ lies in its active throughput interval.

We establish convexity by showing the second derivative is positive:

$$\begin{aligned} \frac{d^2}{d\theta^2} F_J^*(\theta; n) &= \frac{d}{d\theta} \left(\frac{d}{d\theta} F_J(p_s^*(\theta); n) \right) \\ &= \frac{\frac{d}{dp_s^*(\theta)} \frac{d}{d\theta} F_J(p_s^*(\theta); n)}{\frac{d}{dp_s^*} T(p_s^*, 1, n)} \\ &\stackrel{(a)}{=} \frac{\frac{d}{dp_s^*(\theta)} \left(\frac{\frac{d}{dp_s^*(\theta)} F_J(p_s^*(\theta); n)}{\frac{d}{dp_s^*} T(p_s^*, 1, n)} \right)}{\frac{d}{dp_s^*} T(p_s^*, 1, n)} \\ &= \frac{\frac{d^2}{dp_s^*(\theta)^2} F_J(p_s^*(\theta); n) \frac{d}{dp_s^*} T(p_s^*, 1, n) - \frac{d}{dp_s^*(\theta)} F_J(p_s^*(\theta); n) \frac{d^2}{dp_s^{*2}} T(p_s^*, 1, n)}{\left(\frac{d}{dp_s^*} T(p_s^*, 1, n) \right)^3}, \end{aligned} \quad (88)$$

where (a) is from (83). Since we know $\frac{d}{dp_s^*} T(p_s^*, 1, n) < 0$ for $p_s^* \in (0, 1/n)$ (applying Prop. 3, item 1)), showing $\frac{d^2}{d\theta^2} F_J^*(\theta; n) > 0$ is equivalent to showing the numerator in (88) is negative. Thus we compute

$$\begin{aligned} &\frac{d^2}{d\theta^2} F_J^*(\theta; n) \cdot \left(\frac{d}{dp_s^*} T(p_s^*, 1, n) \right)^3 \\ &= - \frac{2(n-1)^2 (np_s^* - 1)^2 (np_s^* + n - 2)^2}{n(n + p_s^* - 2)^4 \left(\frac{n + p_s^* - 2}{n-1} \right)^{-n}} \cdot \frac{f_{\text{nu}}(n; p_s^*)}{f_{\text{de}}(n; p_s^*)^3}, \end{aligned} \quad (89)$$

where the functions f_{nu} and f_{de} are defined in (71) in Lem. 3. Hence we need to show $\frac{f_{\text{nu}}(n; p_s^*)}{f_{\text{de}}(n; p_s^*)^3}$ is positive. This follows from Lem. 3 which assumes $n \geq 3$. For $n = 2$ we can actually prove the convexity directly, leveraging the closed-form expression shown in Prop. 5. Specifically, the second derivative can be computed as

$$\frac{d^2}{d\theta^2} F_J^*(\theta; 2) = \frac{2\theta^2(-2\theta + 3) + 2}{(\theta^2 + 2\theta - 1)^3}, \quad (90)$$

which can be shown to be positive for $\theta \in [1/2, 1)$. This completes the proof. \blacksquare

APPENDIX III

PROOFS FROM §V

Proofs from §V-A, §V-B, and §V-C are given in Appendix III-A, Appendix III-B, and Appendix III-C, respectively.

A. Proofs from §V-A

The following lemma is used in the proof of Prop. 8 for the $\alpha > 1$ case.

Lemma 4: Given $p_s \in (0, 1/n)$, $k \in [n-1]$ and $\alpha \geq 1$, the function $f_2(p_s, k; \alpha)$ defined in (102) is decreasing in α .

Proof: Recall the notation shorthand r_x defined in (14) in §III-B and observe $r_x > 1$. A scaled version of the partial derivative of f_2 w.r.t. α is

$$(\alpha - 1)^2 \frac{\partial}{\partial \alpha} f_2(p_s, k; \alpha) = g_1(p_s, k; \alpha) \quad (91)$$

where

$$g_1(p_s, k; \alpha) \equiv \frac{p_l - p_s}{r_x^\alpha - 1} + \frac{\alpha(\alpha - 1)(p_l - p_s)r_x^\alpha \log r_x}{(r_x^\alpha - 1)^2} - p_s(1 - p_l). \quad (92)$$

We must show $g_1 \leq 0$ for all $\alpha \geq 1$. Towards that goal, the first derivative of g_1 with respect to α is

$$\frac{\partial}{\partial \alpha} g_1(p_s, k; \alpha) = -\frac{(\alpha - 1)(p_l - p_s)r_x^\alpha \log r_x}{(r_x^\alpha - 1)^3} \cdot g_2(p_s, k; \alpha) \quad (93)$$

where

$$g_2(p_s, k; \alpha) \equiv \tilde{g}_2(r_x; \alpha) = -2r_x^\alpha + \alpha(r_x^\alpha + 1) \log r_x + 2, \quad (94)$$

and $\tilde{g}_2(r_x; \alpha)$ is a reparameterization of $g_2(p_s, k; \alpha)$. The derivative of $\tilde{g}_2(r_x; \alpha)$ with respect to α is

$$\frac{\partial}{\partial \alpha} \tilde{g}_2(r_x; \alpha) = \tilde{g}_3(r_x; \alpha) r_x^\alpha \log r_x \quad (95)$$

where

$$\tilde{g}_3(r_x; \alpha) \equiv -1 + r_x^{-\alpha} + \alpha \log r_x. \quad (96)$$

Thus \tilde{g}_3 determines the sign of $\frac{\partial}{\partial \alpha} \tilde{g}_2(r_x; \alpha)$. We can verify $\frac{\partial \tilde{g}_3}{\partial \alpha} = (\alpha + 1) \log r_x > 0$, and furthermore

$$\begin{aligned} \tilde{g}_3(r_x; 1) &= -1 + \frac{1}{r_x} + \log r_x \\ &\geq -\frac{r_x - 1}{r_x} + 2 \left(\frac{r_x - 1}{r_x + 1} \right) \\ &= \frac{(r_x - 1)^2}{r_x(r_x + 1)} > 0. \end{aligned} \quad (97)$$

The inequality comes from a series expansion of the natural logarithm based on the inverse hyperbolic tangent function

$$\log y = 2 \tanh^{-1} \frac{y - 1}{y + 1} = 2 \sum_{n=0}^{\infty} \frac{1}{2n + 1} \left(\frac{y - 1}{y + 1} \right)^{2n+1}, \quad (98)$$

valid for any $y > 0$. This shows $\frac{\partial}{\partial \alpha} \tilde{g}_2(r_x; \alpha) \geq 0$ meaning \tilde{g}_2 is nondecreasing in α . Next,

$$\begin{aligned} \tilde{g}_2(r_x; 1) &= -2r_x + (r_x + 1) \log r_x + 2 \\ &> -2r_x + (r_x + 1) 2 \left(\frac{r_x - 1}{r_x + 1} \right) + 2 = 0, \end{aligned} \quad (99)$$

where the bounding comes again from the series expansion of logarithm based on the inverse hyperbolic tangent function. This means $\tilde{g}_2 \geq 0$ (and in particular, $g_2 \geq 0$) for all $\alpha \geq 1$, which, according to (93), implies g_1 is decreasing in α for $\alpha \geq 1$. Since we can verify $g_1(p_s, k; 1) = 0$, this means $g_1(p_s, k; \alpha) \leq 0$ for all $\alpha \geq 1$. It then follows from (91) that $\frac{\partial f_2}{\partial \alpha} \leq 0$ for all $\alpha \geq 1$. This concludes the proof of the lemma. ■

Proof of Prop. 8: The cases $\alpha > 1$ and $\alpha = 1$ are proved separately. Recall the notation shorthands $r_x, r_{\bar{p}}$ defined in (14) in §III-B and observe $r_x, r_{\bar{p}} > 1$.

Proof for the case $\alpha > 1$.

Fix n' . We will write the objective $F_\alpha(p_s, k, n')$, defined in (28), as $F_\alpha(p_s, k)$ to suppress the dependence on n' . Similar to the proof of Prop. 6, we treat k as a continuous variable and compute the *total derivative* to take into account the throughput constraint. More precisely, we apply (54) and (55) (with F_{-1} replaced by F_α), which yields

$$\frac{d}{dk} F_\alpha(p_s(k), k) = \frac{x_l^{-\alpha} x_s^{1-\alpha}}{p_s(1-n'p_s)(\alpha-1)B} f_1(p_s, k) \quad (100)$$

where

$$\begin{aligned} f_1(p_s, k) &\equiv (\alpha-1)(1-p_s)B(n'-k)(x_l^\alpha - x_s^\alpha) \log r_{\bar{p}} \\ &\quad - (1-n'p_s)((\alpha+p_s-1)Bx_l^\alpha - (1-p_s)(-B+\alpha(n'-k))x_s^\alpha) \end{aligned} \quad (101)$$

and $B = (n'-k)(1-p_l)$. To show $f_1(p_s, k)$ in (100) is nonnegative, we show an equivalent inequality which is less “coupled”. More precisely, showing $f_1(p_s, k)$ is nonnegative is equivalent to showing

$$\log r_{\bar{p}} \geq \frac{1-n'p_s}{(n'-k)(1-p_s)(1-p_l)} f_2(p_s, k; \alpha) \quad (102)$$

where

$$f_2(p_s, k; \alpha) \equiv \frac{(\alpha-1+p_s)(1-p_l)r_x^\alpha - (\alpha-1+p_l)(1-p_s)}{(\alpha-1)(r_x^\alpha - 1)}. \quad (103)$$

Observe in (102), only one side of the inequality involves logarithm and only one side has terms involving α (c.f., showing the positiveness of (56) via (58), in the proof of Prop. 6). In particular, only f_2 depends on α . Since Lem. 4 asserts f_2 is decreasing in α for the regime of interest, this means to prove (102) we only need to prove it for the $\alpha = 1$ case. Applying L'Hôpital's rule, we have

$$\lim_{\alpha \rightarrow 1} f_2(p_s, k; \alpha) = \frac{(1-p_s)(p_l(1+p_s \log r_x) - p_s)(1-p_l)}{p_l - p_s}. \quad (104)$$

Observing $r_x = \frac{p_l}{p_s} + r_{\bar{p}}$ and $\frac{1-n'p_s}{(n'-k)(1-p_s)} = 1$, showing (102) amounts to showing

$$f_3(p_s, k) \equiv (1-p_s p_l) \log r_{\bar{p}} - p_l + p_s - p_s p_l \log \frac{p_l}{p_s} \geq 0. \quad (105)$$

The partial derivative of f_3 w.r.t. k is

$$\frac{\partial}{\partial k} f_3(p_s, k) = -\frac{1-n'p_s}{(n'-k)^3(1-p_l)} f_4(p_s, k) \quad (106)$$

where

$$f_4(p_s, k) \equiv p_s(1-p_l)(n'-k) \log r_x - (1-n'p_s). \quad (107)$$

By applying the logarithm inequality (10), $f_4(p_s, k)$ may be shown to be upper bounded by 0. This implies $f_3(p_s, k)$ is increasing in k , and thus it suffices to verify $f_3(p_s, 0) \geq 0$. Note that although the $k = 0$ case is not included in the $|\mathcal{V}(\mathbf{p})| = 2$ scenario, all the relevant functions are nonetheless well-defined. We must show

$$n' f_3(p_s, 0) = n' p_s + p_s \log(n' p_s) + (n' - p_s) \log \frac{n'(1 - p_s)}{n' - 1} - 1 \geq 0. \quad (108)$$

By taking the partial derivative of $f_3(p_s, 0)$ w.r.t. p_s it is easily seen that it is nonpositive (by checking the monotonicity of $\frac{d}{dp_s} f_3(p_s, 0)$ w.r.t. p_s). Therefore, to show $f_3(p_s, 0) \geq 0$ over $p_s \in [0, 1/n']$ it suffices to show $f_3(1/n', 0) \geq 0$. As $f_3(1/n', 0) = 0$, the above arguments collectively imply $f_3(p_s, k) \geq 0$ for all $0 \leq k \leq n' - 1$ and $p_s \in (1/n', 1)$. Therefore inequality (102) is proved, which means the total derivative (100) is positive and thus the optimal $k^* = n' - 1$.

Proof for the case $\alpha = 1$.

We again fix n' and suppress the dependence of $F_1(p_s, k, n')$ on n' :

$$F_1(p_s, k) = \log \left(p_s^k p_l^{n'-k} \left((1 - p_s)^k (1 - p_l)^{n'-k} \right)^{n'-1} \right). \quad (109)$$

We also write $F_1(p_s(k), k)$ to take into account the throughput constraint. Applying (54) and (55) (with F_{-1} replaced by F_1) yields (where \tanh^{-1} is the inverse hyperbolic tangent function) the *total derivative*

$$\begin{aligned} \frac{d}{dk} F_1(p_s(k), k) &\equiv h_1(p_s, k) \\ &= \frac{1}{p_s(k p_s - 1)} (p_s(1 - k p_s) \log(p_l/(1 - p_l)) - (n' - k) \log r_{\bar{p}} - \\ &\quad 2 p_s(k p_s - 1) \tanh^{-1}(1 - 2 p_s) - n' p_s + 1). \end{aligned} \quad (110)$$

We further compute the partial derivative of $h_1(p_s, k)$ w.r.t. k and get

$$\frac{\partial}{\partial k} h_1(p_s, k) = \frac{-(1 - n' p_s)}{p_s(k p_s - 1)^2 (n' - k - 1 + k p_s)} h_2(p_s, k), \quad (111)$$

where

$$h_2(p_s, k) \equiv (n' - k - 1 + k p_s) \log r_{\bar{p}} + n' p_s - 1. \quad (112)$$

Applying inequality (10), $h_2(p_s, k)$ may be bounded as

$$h_2(p_s, k) \leq \frac{2 k p_s (n' p_s - 1)}{n' - k - 1 + k p_s} < 0, \quad (113)$$

which shows $\frac{\partial}{\partial k} h_1(p_s, k) > 0$. Therefore, to show $h_1(p_s, k) > 0$ for all $p_s \in (0, 1/n')$ and $k \in [n' - 1]$ we only need to show $h_1(p_s, 1) > 0$, or equivalently,

$$p_s(1 - p_s) h_1(p_s, 1) = (n' - 1) \log r_{\bar{p}} - p_s(1 - p_s) \log r_x - (1 - n' p_s) > 0. \quad (114)$$

We rearrange terms and seek to prove the equivalent condition $h_3(p_s, k) > 0$, for

$$h_3(p_s, k) \equiv \frac{(n' - 1) \log r_{\bar{p}} - (1 - n' p_s)}{p_s(1 - p_s)} - \log r_x. \quad (115)$$

Computing the partial derivative w.r.t. p_s and applying inequality (10), we have the upper bound

$$\frac{\partial}{\partial p_s} h_3(p_s, k) \leq -\frac{p_s(1 - n'p_s)^2}{1 - p_s} < 0. \quad (116)$$

This means $h_3(p_s, k)$ is decreasing in p_s and therefore to show $h_3(p_s, k) > 0$ holds for all $p_s \in (0, 1/n')$ it suffices to show $h_3(1/n', k) \geq 0$. We can verify this indeed holds with equality. This completes the proof that $h_1(p_s, k)$, namely the total derivative (110), is positive, implying the optimality of $k^* = n' - 1$. ■

B. Proofs from §V-B

Proof of Thm. 4: Regime i): $\theta \leq \theta_n$. That the maximizer is a uniform vector follows from the Schur-concavity of F_α w.r.t. \mathbf{x} (Prop. 1, or by applying Thm. A. 4 in Ch. 3 of [26]), and the fact that when $\theta \leq \theta_n$ the “all-rates equal” vector is always feasible, as the uniform vector is majorized by all the other vectors that have the same component sums. To establish this feasibility, we assume the optimal rate vector \mathbf{x}^* is such that $x_i^* = p^*(1 - p^*)^{n-1} = \theta/n$, $i \in [n]$, and attempt to solve for $p^* \in [0, 1]$. The existence of such a p^* follows from Lem. 1 and hence the feasibility is proved.

When $\alpha = 1$, an alternative way to show the “all-rates equal” vector is optimal is by using the AM-GM inequality $\tilde{F}_1(\mathbf{x}) = \prod_i x_i \leq (\sum_i x_i/n)^n = (\theta/n)^n$, where $\tilde{F}_1 \equiv e^{F_1(\mathbf{x})}$. As this inequality is tight when all the x_i ’s (p_i ’s) are equal, the maximum $\tilde{F}_1^* = (\theta/n)^n$ will be attained if there exists a vector $\mathbf{p}^* = p^*\mathbf{1}$ that satisfies the throughput constraint namely $x_i^* = p^*(1 - p^*)^{n-1} = \theta/n$, $i \in [n]$.

Regime ii): $\theta \in (\theta_n, 1)$. First, $\mathbf{p}^* \in \partial\mathcal{S}$ follows from Cor. 1 in §III-A. Second, observe $n'^* = n$ as otherwise any inactive user (i.e., one with zero contention probability) will make the objective F_α go to $-\infty$. Third, we claim $\mathbf{p}^* \in \partial\mathcal{S}_2$. To see this, we apply Prop. 4 (item i)), which, together with the fact $\mathbf{p}^* \in \partial\mathcal{S}$ and $n'^* = n$, implies that there is no feasible point if $|\mathcal{V}(\mathbf{p}^*)| = 1$, and hence $|\mathcal{V}(\mathbf{p}^*)| = 2$, meaning $\mathbf{p}^* \in \partial\mathcal{S}_2$. Fourth, when $n'^* = n$ and θ are fixed, the throughput constraint (17) implicitly defines p_s as a function of k and thus we write $F_\alpha(p_s(k), k)$ (with n'^* suppressed). It then follows from the analysis based on the total derivative, shown in Prop. 8, that the optimal $k^* = n - 1$. Finally, the existence and uniqueness of p_s^* follows from Prop. 3 and recognizing that (30) is (17) specialized with $k = n - 1$ and $n' = n$. ■

Proof of Thm. 5: Regime i): $\theta \leq \theta_n$. Denote the original optimization problem (26) with a throughput equality constraint $T(\mathbf{x}) = \hat{\theta}$ by $P_{=}(\hat{\theta})$, and denote the current optimization problem with a throughput inequality constraint $T(\mathbf{x}) \geq \theta$ by $P_{\geq}(\theta)$. The current problem, $P_{\geq}(\theta)$, may be viewed as a two-layer optimization problem where the inner layer is $P_{=}(\hat{\theta})$, i.e., $P_{\geq}(\theta) = \max_{\hat{\theta} \in [\theta, 1]} P_{=}(\hat{\theta})$. This can be further decomposed as the following

$$P_{\geq}(\theta) = \max \left(\max_{\hat{\theta} \in [\theta, \theta_n]} P_{=}(\hat{\theta}), \max_{\hat{\theta} \in (\theta_n, 1)} P_{=}(\hat{\theta}), P_{=}(1) \right). \quad (117)$$

For the first term in (117), since we can verify that $F_\alpha^*(\theta)$ in (29) of Thm. 4 is increasing in θ , at least for regime 1, this shows $\max_{\hat{\theta} \in [\theta, \theta_n]} P_{=}(\hat{\theta}) = P_{=}(\theta_n)$ with the maximizer $\mathbf{p} = (1/n)\mathbf{1}$. For the second term, based on the fact that there exists a tradeoff between target throughput and the α -fair objective for regime 2 (Thm. 6, item 2)), it follows that $\max_{\hat{\theta} \in (\theta_n, 1)} P_{=}(\hat{\theta}) \leq P_{=}(\theta_n)$. For the third term, it can be seen that $P_{=}(1) = -\infty$ because the only

feasible point achieving a target throughput of 1 is \mathbf{e}_i . Therefore, the solution of (117) when $\theta \leq \theta_n$ is given in (31), attained when $\mathbf{p}^* = (1/n)\mathbf{1} = \mathbf{u}$.

Regime *ii*): $\theta \in (\theta_n, 1)$. Observe the maximum of the objective will be attained when there exists some $\theta^* \in [\theta, 1)$ for which the throughput constraint holds with equality namely $T(\mathbf{x}(\mathbf{p}^*)) = \theta^*$, then similar to what was shown in the proof of Thm. 4 (regime 2), we can show $\mathbf{p}^* \in \partial S_2$. Consequently, Prop. 4 (item *ii*)) says the throughput inequality constraint is tight, namely $\theta^* = \theta$, and thus the rest part in the proof of Thm. 4 (regime 2) applies here. Therefore the assertion in regime 2 of Thm. 4 continues to hold. ■

C. Proofs from §V-C

The following lemma is used in the proof of item 2) in Thm. 6.

Lemma 5: Assume $\alpha > 1$ and $n > 2$. The cubic polynomial $f_{\text{cubic}}(p_s)$ in (141) has only one root over $p_s \in (0, 1/n)$.

Proof: First, observe Descartes's rule of sign is not sufficient, as it can only assert there are either one or three positive roots. We instead use the Budan-Fourier Theorem, as was done in the proof of Lem. 3. Specifically, let $v(p)$ denote the number of sign changes (i.e., sign variation) of the Fourier sequence $\{f_{\text{cubic}}(p), f_{\text{cubic}}^{(1)}(p), f_{\text{cubic}}^{(2)}(p), f_{\text{cubic}}^{(3)}(p)\}$ when $p_s = p$. We can show $v(0) = 3$ (the signs of the Fourier sequence are: $- + - +$) and $v(1/n) = 2$ (the signs of the Fourier sequence are: $+ + - +$). Since $v(0) - v(1/n) = 1$, this means the polynomial $f_{\text{cubic}}(p_s)$ only has one root on $(0, 1/n)$.

The following expressions are used to establish the signs of the computed Fourier sequence:

$$\begin{aligned} f_{\text{cubic}}(0) &= 1 - 4\alpha^2 \\ f_{\text{cubic}}^{(1)}(0) &= 2n(4\alpha^2 - 1) \\ f_{\text{cubic}}^{(2)}(0) &= (-10\alpha^2 + 4\alpha + 4)n^2 + (-8\alpha - 6)n + 6 \\ f_{\text{cubic}}^{(3)}(0) &= 6n((\alpha - 1)n + 1)(\alpha n + 1), \end{aligned} \tag{118}$$

and

$$\begin{aligned} f_{\text{cubic}}(1/n) &= (1 - 2p_s)(1 - 2p_s + \alpha) \\ f_{\text{cubic}}^{(1)}(1/n) &= -2\alpha + (\alpha^2 + \alpha + 2)n - 9(1 - p_s) \\ f_{\text{cubic}}^{(2)}(1/n) &= (-4\alpha^2 - 2\alpha + 4)n^2 + (4\alpha - 12)n + 12 \\ f_{\text{cubic}}^{(3)}(1/n) &= 6n((\alpha - 1)n + 1)(\alpha n + 1). \end{aligned} \tag{119}$$

Proof of Thm. 6: We write $F_{\alpha}^*(\theta; n)$ to emphasize θ is the free variable and (α, n) are viewed as parameters. The feasible set Λ is parameterized by \mathbf{p} via (2) and when $\theta \in (\theta_n, 1)$ we know from Thm. 4 that the unique extremizer can be characterized by the tuple (p_s^*, k^*, n'^*) as functions of θ . Therefore the notation $F_{\alpha}^*(\theta; n)$ should

be understood as

$$\begin{aligned}
F_\alpha^*(\theta; n) &\equiv F_\alpha(\mathbf{x}(\mathbf{p}(p_s^*(\theta), k^*(\theta), n'^*(\theta))); n) \\
&= F_\alpha(\mathbf{x}(\mathbf{p}(p_s^*(\theta), n-1, n)); n) \\
&\equiv F_\alpha(p_s^*(\theta); n),
\end{aligned} \tag{120}$$

where the second equality follows from Thm. 4, and the third equivalence is a shorthand notation.

Observe also the identity (c.f., (76))

$$\theta \equiv T(p_s^*(\theta), n-1, n), \tag{121}$$

for $T(p_s, k, n')$ defined in (15) and p_s^* solved from (30). This is useful for computing the dependence of p_s^* on θ .

Item 1). We first look at the monotonicity of $p_s^*(\theta)$, $p_l^*(\theta)$ in θ . That $p_s^*(\theta)$ is decreasing in θ follows from

$$\frac{dp_s^*(\theta)}{d\theta} = \left(\frac{d\theta(p_s^*)}{dp_s^*} \right)^{-1} = \left(\frac{d}{dp_s^*} T(p_s^*, n-1, n) \right)^{-1} < 0, \tag{122}$$

where we apply (121) and the negativity follows from Prop. 3 (item 1)) in §III-B. That $p_l^*(\theta) = p_l(p_s^*(\theta), n-1, n)$ is increasing in θ follows easily from the definition of p_l in Def. 1 and (122).

Next, we look at the dependence of $x_s^*(\theta)$, $x_l^*(\theta)$ upon θ . We have

$$\begin{aligned}
x_s^* &= (n-1)p_s^{*2}(1-p_s^*)^{n-2} \\
x_l^* &= p_l^*(1-p_s^*)^{n-1}.
\end{aligned} \tag{123}$$

That $\frac{dx_l^*(\theta)}{d\theta} > 0$ follows from $\frac{dp_s^*(\theta)}{d\theta} < 0$ and $\frac{dp_l^*(\theta)}{d\theta} > 0$. To show $\frac{dx_s^*(\theta)}{d\theta} < 0$, applying the chain rule and recalling (122), we need to show $p_s^{*2}(1-p_s^*)^{n-2}$ is increasing in p_s^* . Since we can verify the function $p^2(1-p)^{n-2}$ is increasing for all $p \in (0, 2/n) \supseteq (0, 1/n)$, this proves $\frac{dx_s^*(\theta)}{d\theta} < 0$.

Item 2). First, we prove *i*) monotonicity, *ii*) continuity, and *iii*) differentiability. Towards *i*) monotonicity, we compute:

$$\begin{aligned}
\frac{d}{d\theta} F_\alpha^*(\theta; n) &= \frac{d}{dp_s^*(\theta)} F_\alpha(p_s^*(\theta); n) \left(\frac{d\theta(p_s^*)}{dp_s^*} \right)^{-1} \\
&= \frac{d}{dp_s^*(\theta)} F_\alpha(p_s^*(\theta); n) \left(\frac{d}{dp_s^*} T(p_s^*, k^*, n'^*) \right)^{-1},
\end{aligned} \tag{124}$$

where the second equality comes from (121). When $\alpha = 1$ and $\alpha > 1$, we get

$$\begin{aligned}
\frac{d}{d\theta} F_1^*(\theta; n) &= -\frac{1-p_s^*}{p_s^*} \frac{1}{x_l^*} < 0, \\
\frac{d}{d\theta} F_{\alpha>1}^*(\theta; n) &= -\frac{(1-p_l^*)x_s^{*- \alpha} - (1-p_s^*)x_l^{*- \alpha}}{1-n'^*p_s^*} (n'^* - k^*)
\end{aligned} \tag{125}$$

We will show (125) is negative for all $p_s^* \in (0, 1/n')$. Namely we want to show $(1-p_l^*)x_s^{*- \alpha} - (1-p_s^*)x_l^{*- \alpha} > 0$, which is equivalent to $\frac{p_s^{*- \alpha}}{(1-p_s^*)^{1-\alpha}} > \frac{p_l^{*- \alpha}}{(1-p_l^*)^{1-\alpha}}$. Thus it suffices to show

$$h(z) \equiv \frac{z^{-\alpha}}{(1-z)^{1-\alpha}} \tag{126}$$

with $\alpha > 1$ is decreasing in z for $z \in (0, 1)$. Since

$$\frac{dh(z)}{dz} = z^{-\alpha-1}(1-z)^{\alpha-2}(z-\alpha) < 0 \quad (127)$$

this shows the desired monotonicity of $h(z)$, establishing that the optimal α -fair objective is decreasing in θ .

Next, we prove *ii*) continuity. *a*) since the roots (on the complex plane) of a polynomial equation are continuous in its coefficients [29, §3.9] and since the polynomial equation (30) only has a single real root (p_s^*) it must be continuous also. *b*) the function F_α^* in (120) is continuous in p_s^* .

Third, we prove *iii*) differentiability. This follows from the fact that the derivatives given in (125) are continuous in p_s^* , which are themselves continuous in θ , c.f., (122).

Next, we investigate convexity (concavity).

Similar to what was done in the proof of Thm. 3 (item 5)), we compute the second derivative and investigate its sign:

$$\begin{aligned} \frac{d^2}{d\theta^2} F_\alpha^*(\theta; n) &= \frac{d}{d\theta} \left(\frac{d}{d\theta} F_\alpha(p_s^*(\theta); n) \right) \\ &\stackrel{(b)}{=} \frac{\frac{d}{dp_s^*(\theta)} \left(\frac{d}{d\theta} F_\alpha(p_s^*(\theta); n) \right)}{\frac{d}{dp_s^*} T(p_s^*, n-1, n)} \\ &\stackrel{(c)}{=} \frac{\frac{d}{dp_s^*(\theta)} \left(\frac{\frac{d}{dp_s^*(\theta)} F_\alpha(p_s^*(\theta); n)}{\frac{d}{dp_s^*} T(p_s^*, n-1, n)} \right)}{\frac{d}{dp_s^*} T(p_s^*, n-1, n)} \\ &= \frac{\frac{d^2}{dp_s^*(\theta)^2} F_\alpha(p_s^*(\theta); n) \frac{d}{dp_s^*} T(p_s^*, n-1, n) - \frac{d}{dp_s^*(\theta)} F_\alpha(p_s^*(\theta); n) \frac{d^2}{dp_s^{*2}} T(p_s^*, n-1, n)}{\left(\frac{d}{dp_s^*} T(p_s^*, n-1, n) \right)^3}, \quad (128) \end{aligned}$$

where (b) is due to the chain rule and (121) and (c) is from (124). Since we know from Prop. 3 (item 1)) $\frac{d}{dp_s^*} T(p_s^*, n-1, n) < 0$ for $p_s^* \in (0, 1/n)$, showing $\frac{d^2}{d\theta^2} F_\alpha^*(\theta; n) \geq 0$ is equivalent to showing the numerator in (128) is negative / positive.

We consider the $\alpha = 1$ and $\alpha \geq 0$ but $\alpha \neq 1$ cases separately. First, for $\alpha = 1$:

$$\frac{d^2}{d\theta^2} F_1^*(\theta; n) \left(\frac{d}{dp_s^*} T(p_s^*, n-1, n) \right)^3 = \frac{(n-1)^2(1-p_s^*)^{n-5}(np_s^*-2)^2(np_s^*-1)^2}{p_s^{*2}(-np_s^*+p_s^*+1)^2} f_{\text{quad}}(p_s^*, n) \quad (129)$$

where

$$f_{\text{quad}}(p_s^*, n) \equiv (n^2 - n)p_s^{*2} + (3 - 3n)p_s^* + 1 \quad (130)$$

Next, for $\alpha \geq 0$ but $\alpha \neq 1$:

$$\frac{d^2}{d\theta^2} F_\alpha^*(\theta; n) \left(\frac{d}{dp_s^*} T(p_s^*, n-1, n) \right)^3 = -(n-1)^3(1-p_s^*)^{2n-7}(np_s^*-2)^2 x_s^{*- \alpha} f_2(p_s^*, \alpha, n) \quad (131)$$

where

$$f_2(p_s^*, \alpha, n) \equiv -Z(p_s^*, \alpha, n) + (1-p_s^*) - \left(\frac{x_s^*}{x_l^*} \right)^\alpha (1-p_s^*) \left(\frac{Z(p_s^*, \alpha, n)}{1-(n-1)p_s^*} + 1 \right), \quad (132)$$

and

$$Z(p_s^*, \alpha, n) \equiv \alpha(np_s^*-2)(np_s^*-1). \quad (133)$$

Recall α and n are assumed to be fixed. We may sometimes drop them from the parameter list by e.g., writing $f_2(p_s^*, \alpha, n)$ as $f_2(p_s^*)$.

Case 1: $\alpha > 1$. In this case, since the sign of $\frac{d^2}{d\theta^2} F_\alpha^*(\theta; n)$ equals the sign of $f_2(p_s^*, \alpha, n)$, our goal is to show, as p_s^* increases from 0 to $1/n$:

- when $n > 2$, there exists a thresholding $\tilde{p}_s^* = \tilde{p}_s^*(\alpha, n)$ below (above) which $f_2 < (>) 0$, corresponding to the T-F curve being concave when θ is large (convex when θ is small);
- when $n = 2$, it always holds that $f_2 < 0$, meaning the T-F curve is always concave.

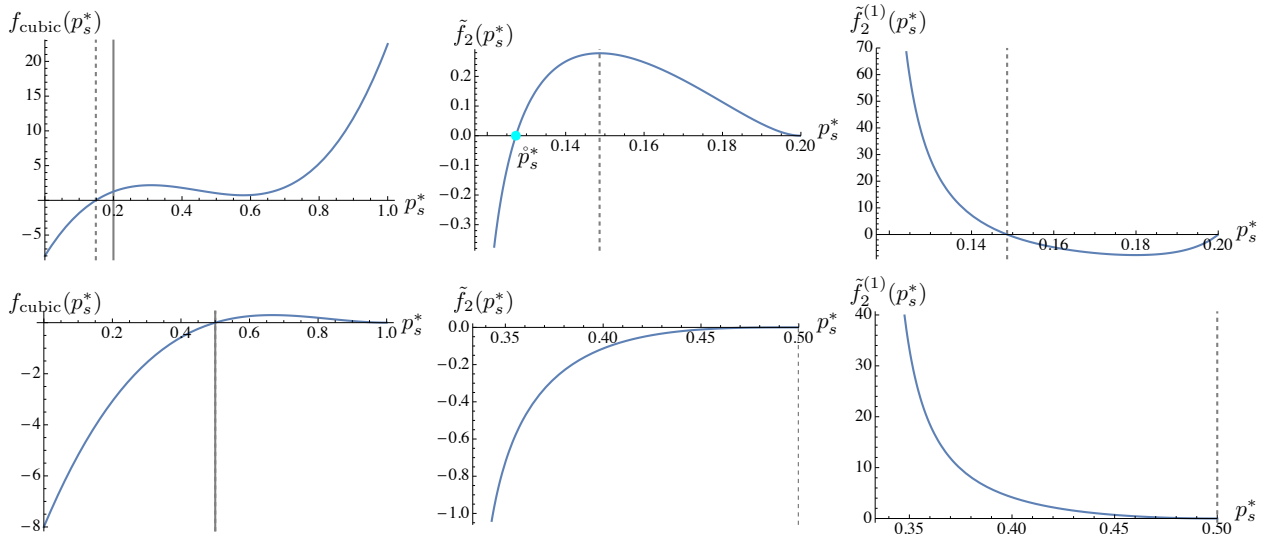


Fig. 11. Illustration of the proof of item 2) in Thm. 6 regarding the thresholding p_s^* . Shown are the polynomial $f_{\text{cubic}}(p_s^*)$ (141) (1st column), the function $\tilde{f}_2(p_s^*)$ (2nd column), and its first derivative w.r.t. p_s^* denoted $\tilde{f}_2^{(1)}(p_s^*)$ (3rd column), for $n = 5$ (top row) and $n = 2$ (bottom row) respectively. In both cases $\alpha = 1.5$. The solid gridlines indicate $1/n$; the dashed gridlines indicate the only stationary point of \tilde{f}_2 on $(0, 1/n)$ (also the unique real root of f_{cubic} on $(0, 1/n)$). Except 1st column, the plot ranges for the horizontal axis p_s^* are $(p_{s-}^*, 1/n)$. For $n = 5$, $p_{s-}^* \approx 0.11683$ and the stationary point of \tilde{f}_2 is at $p_s^* \approx 0.1487$; for $n = 2$, $p_{s-}^* \approx 0.3333$ and the stationary point of \tilde{f}_2 is at $p_s^* = 0.5$. The thresholding $\tilde{p}_s^* \in (0, 1/n)$ exists only when $n > 2$. Here when $n = 5$ and $\alpha = 1.5$, we have $\tilde{p}_s^* \approx 0.1273$ solved from (142) and marked as the cyan dot in the top-middle figure where $\tilde{f}_2(p_s^*)$ versus p_s^* is shown.

Subcase 1: $n > 2$

Directly showing the desired monotonicity (change) of f_2 w.r.t. p_s^* does not seem easy, as the derivative of f_2 w.r.t. p_s^* has polynomials of p_s^* further raised to the power of α . Therefore we seek to show the following equivalent condition (recall $p_s^* \in (0, 1/n)$ and observe $Z \geq 0$).

$$f_2 \leq 0 \iff \frac{-Z + (1 - p_s^*)}{(1 - p_s^*) \left(\frac{Z}{1 - (n-1)p_s^*} + 1 \right)} \leq \left(\frac{x_s^*}{x_l^*} \right)^\alpha \quad (134)$$

First consider the $f_2 < 0$ case. Define the following events

$$\begin{aligned} E_2 &= \{p_s^* \in (0, 1/n) : f_2(p_s^*) < 0\}, \\ \tilde{E}_2 &= \{p_s^* \in (0, 1/n) : \tilde{f}_2(p_s^*) < 0\}, \\ E_Z &= \{p_s^* \in (0, 1/n) : -Z(p_s^*) + (1 - p_s^*) \leq 0\}, \\ \overline{E}_Z &= \{p_s^* \in (0, 1/n) : -Z(p_s^*) + (1 - p_s^*) > 0\}, \end{aligned} \quad (135)$$

where

$$\tilde{f}_2(p_s^*) \equiv \frac{1}{\alpha} \log \left(\frac{-Z + (1 - p_s^*)}{(1 - p_s^*) \left(\frac{Z}{1 - (n-1)p_s^*} + 1 \right)} \right) - \log \frac{x_s^*}{x_l^*}. \quad (136)$$

Observe the following equivalence of events

$$E_2 = E_Z \cup (\overline{E}_Z \cap \tilde{E}_2). \quad (137)$$

By substituting the definition of Z given in (130), the expression $-Z + (1 - p_s^*)$ can be expressed as a quadratic in p_s^* (with a negative coefficient of the term p_s^{*2}) whose smaller (p_{s-}^*) and larger (p_{s+}^*) roots are

$$p_{s\mp}^* = \frac{3\alpha n - 1 \mp \sqrt{\alpha n(\alpha n + 4n - 6) + 1}}{2\alpha n^2}. \quad (138)$$

Therefore (137) is equivalent to

$$E_2 = E'_Z \cup (\overline{E}'_Z \cap \tilde{E}_2), \quad (139)$$

where

$$E'_Z = \{p_s^* \in (0, p_{s-}^*)\}, \quad \overline{E}'_Z = \{p_s^* \in (p_{s-}^*, 1/n)\}, \quad (140)$$

because we can verify that $p_{s+}^* > \frac{3\alpha n - 1}{2\alpha n^2} > \frac{1}{n}$ for $\alpha \geq 1, n \geq 2$, and that $E'_Z = E_Z, \overline{E}'_Z = \overline{E}_Z$.

So we focus on the events $(\overline{E}'_Z \cap \tilde{E}_2)$ in (139). Our goal now is to show there exists one and only one thresholding $\hat{p}_s^* \in (p_{s-}^*, 1/n)$ upon which \tilde{f}_2 (and hence f_2 , as implied by (139)) changes its sign. We compute $\frac{\partial \tilde{f}_2}{\partial p_s^*}$ and find the stationary point(s) of \tilde{f}_2 is (are) the root(s) of the cubic equation

$$\begin{aligned} f_{\text{cubic}}(p_s^*) &\equiv p_s^{*3} (\alpha^2 n^3 - \alpha n^3 + 2\alpha n^2 - n^2 + n) + \\ &\quad p_s^{*2} (-5\alpha^2 n^2 + 2\alpha n^2 + 2n^2 - 4\alpha n - 3n + 3) + \\ &\quad p_s^* (8\alpha^2 n - 2n) + 1 - 4\alpha^2. \end{aligned} \quad (141)$$

Lem. 5 shows this cubic equation has only one root on $(0, 1/n)$. It follows that \tilde{f}_2 has only one root on $(p_{s-}^*, 1/n)$. To see this, first note \tilde{f}_2 cannot have any root on $(0, p_{s-}^*]$ as otherwise $f_2 < 0$ wouldn't hold for all $p_s^* \in (0, p_{s-}^*]$, contradicting (139). Second, for the interval $(p_{s-}^*, 1/n)$, we prove by contradiction: assuming \tilde{f}_2 has two or more roots on $(p_{s-}^*, 1/n)$, since it can be verified that $\tilde{f}_2(1/n) = 0$, due to the continuity of \tilde{f}_2 and $\frac{\partial \tilde{f}_2}{\partial p_s^*}$, \tilde{f}_2 must necessarily have at least two stationary points on $(p_{s-}^*, 1/n)$ meaning f_{cubic} defined in (141) has at least two roots on $(p_{s-}^*, 1/n) \subseteq (0, 1/n)$ contradicting Lem. 5. In fact since it can be further verified that the second derivative of \tilde{f}_2 w.r.t. p_s^* at $p_s^* = 1/n$ evaluates to a positive number $(1 - 2p_s^*)(1 - 2p_s^* + \alpha) / ((1 - p_s^*)^2 p_s^{*4})$ meaning \tilde{f}_2 has

a local minimum at $p_s^* = 1/n$: this implies if \tilde{f}_2 has more than one root on $(p_{s-}^*, 1/n)$, then it must necessarily have at least three roots on this interval as $\lim_{p_s^* \rightarrow p_{s-}^*} \tilde{f}_2 = -\infty$. See Fig. 11 (top column) for an illustration.

The thresholding $\tilde{p}_s^*(\alpha > 1, n > 2)$ upon which \tilde{f}_2 changes its sign (namely the root of $\tilde{f}_2(p_s^*)$) can be obtained by solving the following polynomial equation in $p_s^* \in (p_{s-}^*, \frac{1}{n})$:

$$-\alpha(np_s^*-2)(np_s^*-1)+(1-p_s^*)-\left(\frac{(n-1)p_s^{*2}}{(1-p_s^*)(1-(n-1)p_s^*)}\right)^\alpha (1-p_s^*)\left(\frac{\alpha(np_s^*-2)(np_s^*-1)}{1-(n-1)p_s^*}+1\right)=0. \quad (142)$$

This follows by substituting the definition of the terms given in the event $(\bar{E}_Z \cap \tilde{E}_2)$ in (137) and recalling (134) and (129).

Finally, to obtain the thresholding $\hat{\theta}_\alpha(n)$, we employ (121) which yields

$$\hat{\theta}_\alpha(n) = T(p_s^*(\alpha, n), n-1, n), \quad (143)$$

for $T(p_s, k, n')$ defined in (15).

Subcase 2: $n = 2$

In this case, we will show that the T-F curve is concave decreasing for all $p_s^* \in (0, 1/n)$. First, notice both \tilde{f}_2 and $\frac{d\tilde{f}_2}{d\alpha}$ simplify

$$\begin{aligned} \tilde{f}_2 &= \frac{1}{\alpha} \log \left(\frac{2}{1+2\alpha-4\alpha p_s^*} - 1 \right) - 2 \log \left(\frac{p_s^*}{1-p_s^*} \right), \\ \frac{d\tilde{f}_2}{d\alpha} &= -\frac{2(2\alpha-1)(2\alpha+1)(1-2p_s^*)^2}{(1-p_s^*)p_s^*(-1-2\alpha+4\alpha p_s^*)(1-2\alpha+4\alpha p_s^*)}. \end{aligned} \quad (144)$$

Furthermore, since the smaller root p_{s-}^* simplifies to $\frac{1}{4}(2 - \frac{1}{\alpha})$, it can be verified that $\frac{d\tilde{f}_2}{d\alpha}$ is continuous and positive on $(p_{s-}^*, 1/2)$ and $\frac{d}{d\alpha} \tilde{f}_2(1/2) = 0$, $\lim_{p_s^* \rightarrow p_{s-}^*} \frac{d\tilde{f}_2}{d\alpha} = \infty$. This means \tilde{f}_2 is increasing on $p_s^* \in (p_{s-}^*, 1/2)$, from $-\infty$ (when $p_s^* \rightarrow p_{s-}^*$) to 0 (when $p_s^* = 1/2$). Also see Fig. 11 (bottom column) for an illustration. Therefore, according to (139) and by recalling (129) through (137), it means when $n = 2$, the T-F curve is concave for all $p_s^* \in (0, 1/n)$.

Case 2: $\alpha = 1$

In this case, according to (129), the sign of $\frac{d^2}{d\theta^2} F_\alpha^*(\theta; n)$ is opposite to the sign of $f_{\text{quad}}(p_s^*, \alpha, n)$. The symmetry axis of $f_{\text{quad}}(p_s^*, \alpha, n)$ is given by $\frac{3}{2n}$ and it is decreasing on $p_s^* \in (0, 1/n)$ from 1 to $2/n - 1$. Therefore, when $n = 2$, $f_{\text{quad}}(p_s^*, \alpha, n)$ remains positive for all $p_s^* \in (0, 1/n)$ meaning the T-F curve is always concave. When $n > 2$, the thresholding \tilde{p}_s^* is the smaller root of this quadratic namely

$$\tilde{p}_s^*(1, n > 2) = \frac{1}{2n} \left(3 - \sqrt{\frac{5n-9}{n-1}} \right), \quad (145)$$

which can be verified to be the same as obtained by solving (142) with $\alpha = 1$.

Item 3). We now investigate the dependence on n while holding $\alpha \geq 1$ and target throughput $\theta \in (\theta_n, 1)$ both fixed. In this case it is clear from Thm. 4 that F_α^* in (120) should be understood as

$$F_\alpha(p_s^*(n), n) \equiv F_\alpha(\mathbf{x}(\mathbf{p}(p_s^*(n), n-1, n)); n). \quad (146)$$

In the following we compute the *total derivative* of $F_\alpha(p_s^*(n), n)$ w.r.t. n and show it is negative.

$$\begin{aligned} \frac{d}{dn} F_\alpha(p_s^*, n) &= \frac{\partial}{\partial p_s^*} F_\alpha(p_s^*, n) \frac{dp_s^*(n)}{dn} + \frac{\partial}{\partial n} F_\alpha(p_s^*, n) \\ &\stackrel{(a)}{=} \frac{\partial}{\partial p_s^*} F_\alpha(p_s^*, n) \left(-\frac{\frac{\partial}{\partial n} T(p_s^*, n-1, n)}{\frac{\partial}{\partial p_s^*} T(p_s^*, n-1, n)} \right) + \frac{\partial}{\partial n} F_\alpha(p_s^*, n), \end{aligned} \quad (147)$$

where (a) is by using the implicit function theorem, analogous to (54). We now address the cases $\alpha = 1$ and $\alpha > 1$ respectively. When $\alpha = 1$ (147) simplifies to

$$\begin{aligned} \frac{dF_1}{dn} &= \log x_s^* + \frac{(n-1)p_s^{*2} + \log(1-p_s^*)}{p_s^*(1-(n-1)p_s^*)} \\ &< \log x_s^* - 1 < 0, \end{aligned} \quad (148)$$

where the first bounding is by applying (10) to $\log(1-p_s^*)$. When $\alpha > 1$ (147) can be shown to be

$$\frac{dF_{\alpha>1}}{dn} = -\frac{x_s^{*(1-\alpha)} x_l^{*(-\alpha)}}{(\alpha-1)p_s^*(1-np_s^*)} f_1(p_s^*, n) \quad (149)$$

where

$$f_1(p_s^*, n) \equiv (\alpha-1)(1-p_s^*)(p_s^* + \log(1-p_s^*))(x_s^{*\alpha} - x_l^{*\alpha}) + \alpha(1-np_s^*)p_s^* x_l^{*\alpha}. \quad (150)$$

As $p_s^* \in (0, 1/n)$, it follows that

$$p_s^* + \log(1-p_s^*) < p_s^* + (-p_s^*) = 0. \quad (151)$$

As $x_s^{*\alpha} < x_l^{*\alpha}$, it follows that $f_1(p_s^*, n)$ is the summation of two positive numbers and hence $\frac{dF_{\alpha>1}}{dn} < 0$. ■